

# PREFERENTIAL ATTACHMENT RANDOM GRAPHS WITH EDGE-STEP FUNCTIONS

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**ABSTRACT.** We propose a random graph model with preferential attachment rule and *edge-step functions* and prove several properties about it. That is, we consider a random graph model in which at time  $t$ , a new vertex is added with probability  $f(t)$  or a connection between already existing vertices is created with probability  $1 - f(t)$ . All the connections are made according to the preferential attachment rule. To this function of time,  $f$ , we give the name *edge-step function*. We investigate the effect of the edge-step function on the topology of the graphs generated by this preferential attachment scheme. Regarding the degree distribution, we prove that the model may generate graphs following power-law distribution with exponent  $\beta \in (1, 3]$  or graphs having dense complete subgraphs. Moreover, we study how these functions may act on the way of breaking the diameter growth. In particular, we prove that for  $f$  belonging to a specific class, the diameter of the random graph is bounded by a constant, though the order of the graph goes to infinity.

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## 1. INTRODUCTION

In the late 1990s the seminal works of Strogatz and Watts [14] and of Álbért and Barabási [2] brought to light two common features shared by real-life networks: *small diameter* and *power-law* degree distribution. In the first work, the authors observed that large-scale networks of biological, social and technological origins presented diameter of much smaller order than the order of the entire network, a phenomenon they called *small-world*. In the second paper, the authors noted that the fraction of nodes having degree  $k$  decays roughly as  $k^{-\beta}$  for some  $\beta > 0$ , a feature is also known as *scale-freeness*.

These findings motivated the task of proposing and investigating random graph models capable of capturing the two aforementioned features as well as other properties, such as large clique number [1] and maximum degree [12]. The interested reader may be directed to [4, 7, 15] for a summary of rigorous results for many different models.

Usually the models proposed over the years are inductive, in the sense that at each step  $t$  one obtain the random graph  $G_t$  by performing some stochastic operation on  $G_{t-1}$ . In the well known Barabási - Álbért model [2], the stochastic operation consists of at each

step a new vertex being added and a neighbor to it chosen among the previous vertices with probability proportional to its degree. This simple attachment rule, which is known as *preferential attachment*, or PA-rule for short, is capable of producing graphs obeying a power-law distribution with exponent  $\beta = 3$ . Many variants of the original B-A preferential attachment [4, 10, 11] have been introduced. These models also are capable of exhibiting power-law distribution with different values of  $\beta$  and small-world phenomenon.

In the vast majority of these models the vertex set of  $G_t$  grows linearly with  $t$ . Since we know that in many real-world networks the rate of new born nodes decreases with time, e.g., Facebook and other social medias, we propose a new model which exhibits sub-linear growth and also presents power-law degree distribution and small diameter.

The remainder of this Section is organized as follows: in the next subsection we briefly introduce our model. Then we give a discussion of power-law degree distributions with exponent in  $(1, 2]$ , the connection between diameter growth and the vertex set growth and how our results fit in both contexts. We also give a summary of the main techniques applied in the proof of our main results. Finally, we end this introduction describing the paper's organization and introducing some notations and conventions.

**1.1. The Preferential Attachment Scheme with an edge-step function.** The model we propose here has one parameter: a real non-negative function  $f$  with domain given by  $\mathbb{N}$  such that  $\|f\|_\infty \leq 1$ . For the sake of simplicity, we start the process from an initial graph  $G_1$  which is taken to be the graph with one vertex and one loop. We consider the two stochastic operations below that can be performed on any graph  $G$ :

- *Vertex-step* - Add a new vertex  $v$  and add an edge  $\{u, v\}$  by choosing  $u \in G$  with probability proportional to its degree. More formally, conditionally on  $G$ , the probability of attaching  $v$  to  $u \in G$  is given by

$$P(v \rightarrow u|G) = \frac{\text{degree}(u)}{\sum_{w \in G} \text{degree}(w)}.$$

- *Edge-step* - Add a new edge  $\{u_1, u_2\}$  by independently choosing vertices  $u_1, u_2 \in G$  according to the same rule described in the vertex-step. We note that both loops and parallel edges are allowed.

We consider a sequence  $(Z_t)_{t \geq 1}$  of independent random variables such that  $Z_t \stackrel{d}{=} \text{Ber}(f(t))$ . We then define inductively a random graph process  $(G_t(f))_{t \geq 1}$  as follows: start with  $G_1$ . Given  $G_t(f)$ , obtain  $G_{t+1}(f)$  by either performing a *vertex-step* on  $G_t(f)$  when  $Z_t = 1$  or performing an *edge-step* on  $G_t(f)$  when  $Z_t = 0$ .

We will call the function  $f(t)$  by *edge-step function*, though we follow an edge-step at time  $t$  with probability  $1 - f(t)$ . We will also reserve special notation for some particular cases of  $f(t)$ . Given  $M \in \mathbb{Z}_+$  and  $\gamma \in (0, 1]$ , we define the edge step functions  $\ell_M$  and  $q_\gamma$  so that  $\ell_M(t) = (\log(t))^{-M}$  and  $q_\gamma(t) = t^{-\gamma}$ . We also make an abuse of notation and let  $p$

denote both a number in  $(0, 1]$  and a function from  $\mathbb{N}$  to  $[0, 1]$  that is constant and equal to  $p$ .

**1.2. Power-law distribution and continuity on the parameters.** The edge-step functions introduced in this work allow good control over the shape of the power-law degree distribution. In the particular case where  $f(t) \equiv p$ , with  $p \in (0, 1)$ , studied in [4], the functions provided a control of the tail of the power-law distribution producing graphs obeying such laws with a tunable exponent  $\beta = 2 + \frac{p}{2-p}$ . To achieve the exponent  $\beta = 2$  one may expect some sort of continuity on  $p$ : taking the parameter as small as one desires should be enough to lift the power-law distribution's tail so that its rate of decay becomes quadratic. Instead of taking the limit on  $p$ , which has no meaning at first glance, we choose  $f(t)$  to be a decreasing function such that  $f(t) \rightarrow 0$  as  $t \rightarrow \infty$ . In this direction, we prove that for edge-step functions going to zero with logarithmic speed, the exponent  $\beta = 2$  is achieved. However, for the choice of edge-step functions decaying at polynomial speed, the tail is lifted even more and a power-law distribution of exponent smaller than two is obtained. These results are summarized on the Theorem below:

**Theorem 1** (Power-law distribution).  *$G_t(\ell_M)$  obeys a power-law degree distribution with exponent  $\beta = 2$ . On the other hand, If  $\gamma < 1$  then  $G_t(q_\gamma)$  obeys a power-law degree distribution with exponent  $\beta = 2 - \gamma$ .*

We may keep pushing the boundaries further by choosing edge-step functions whose decay is even faster than  $t^{-\gamma}$ , for  $\gamma \in (0, 1)$ . However, setting  $\gamma = 1$  is enough to break the degree distribution down in the sense that all its mass is transported to its tail. In this case, the edge-step function goes fast enough to 0 so that the number of edge-steps taken is high enough to increase the degree of almost all vertices in the graph. In the end, the result is that the proportion of the vertices whose degree is any fixed constant goes to zero as time goes to infinity. Formally, we prove the following result

**Theorem 2.** *Let  $N_t(d)$  be the number of vertices whose degree is  $d$  in  $G_t(q_1)$ . Then, for all fixed  $d \in \mathbb{N}$ , we have*

$$(1.1) \quad \frac{\mathbb{E}[N_t(d)]}{\log(t)} \xrightarrow{t \rightarrow \infty} 0.$$

For another attempt at producing PA-models having  $\beta$  in  $(1, 2]$ , see [5], where the authors propose a model in which the number of edges added at each step is given by a sequence of independent random variables. This new rule is capable of reducing  $\beta$  but the vertex set still grows linearly in time, a property we would like to avoid in this paper.

To better understand how different choices for the edge-step function can shape other properties of the graphs produced by the model, we also introduce a process that couples together every  $(G_t(f))_{t \geq 1}$  for every possible edge-step function  $f$ . This grand coupling has nice monotonicity properties that help us extend previously known results to this new class of models

and rigorously prove that these functions act on the geometry of the graphs by shortening their diameters and increasing their clique number. Concerning the existence of large complete subgraphs, the particular case  $f(t) \equiv p$  produces graphs containing *a.a.s* a complete subgraph of order  $t^{(1-\varepsilon)(1-p)/(2-p)}$  [1] for all  $\varepsilon$  small enough. Again, one may expect to be able to obtain a clique of order close to  $\sqrt{t}$  by choosing  $p$  close to zero. In this direction, the grand coupling helps us prove the following result:

**Theorem 3** (Large cliques). *Let  $f : \mathbb{N} \rightarrow [0, 1]$  be a non-increasing function such that  $f(k) \geq k^{-1}$  and  $\lim_{k \rightarrow \infty} f(k) = 0$ . Define  $F(t) := \sum_{k=1}^t f(k)$ . Then, for every  $\delta \in (0, 1)$ ,  $G_t(f)$  contains a complete subgraph of size  $F(t^{(1-\delta)/2})$  asymptotically almost surely.*

We show that, with high probability, the number of vertices of  $G_t(f)$  is highly concentrated around  $F(t)$ . In particular, the number of vertices in  $G_t(\ell_M)$  has order  $(t/(\log(t))^M)$  and the number of vertices of  $G_t(q_\gamma)$  has order  $t^{1-\gamma}$ , for  $\gamma < 1$ . Therefore, in these cases we can interpret the above result as showing the existence of a large complete subgraph with the same order as the number of vertices of  $G_t(f)$  to a power as close to  $1/2$  as one desires.

This has the flavor of the continuity of observables discussed above. In fact, the process  $(G_t(\ell_M))$  behaves as the limit as  $p \rightarrow 0$  in the following sense:

**Corollary 1.2.** *Let  $\omega(G)$  denote the size of the largest complete subgraph of the graph  $G$ . Then*

$$(1.3) \quad \frac{\log(\omega(G_t(\ell_M)))}{\log(t)} \xrightarrow{t \rightarrow \infty} \frac{1}{2} \text{ a.a.s.}$$

This follows from the above theorem and a deterministic bound on the number of triangles of a given simple graph as a function of its number of vertices and edges [13].

Concerning the case  $G_t(q_1)$ , Theorem 3 implies that in this very particular case, almost half of the vertices lie in a clique, i.e.,  $G_t(q_1)$  contains a dense complete subgraph.

**1.3. Breaking the growth of the diameter.** In order to slow the growth of the diameter of PA-models, two observables play important roles: the maximum degree and the proportion of vertices with low degree. The former tends to concentrate connections on vertices with very high degree acting in the way of shortening the diameter. Whereas the latter, acts in the opposite way. In [16] and [8], the authors have shown that in the configuration model with power-law distribution the diameter order is extremely sensitive to the proportion of vertices with degree 1 and 2.

One way to reduce the effect of low degree vertices on the diameter is via *affine* preferential attachment rules, i.e., introducing a parameter  $\delta$  and choosing vertices with probability proportional to their degree plus  $\delta$ . In symbols, conditionally on  $G_t$ , we connect a new

vertex  $v_{t+1}$  to an existing one  $u$  with probability

$$\mathbb{P}(v_{t+1} \rightarrow u | G_t) = \frac{\text{degree}(u) + \delta}{\sum_{w \in G_t} (\text{degree}(w) + \delta)}.$$

By taking a negative  $\delta$ , the above rule increases the influence of high degree vertices and indeed decreases drastically the diameter's order. For instance, for  $\delta$  positive the diameter of  $G_t$  is at least  $\log(t)$ , whereas for  $\delta < 0$  the diameter of  $G_t$  is at most  $\log(\log(t))$ . See [6] for several results on the diameter of different combinations for the affine preferential attachment rule.

Reducing the effect of low degree vertices is not enough to break the growth of the diameter. The reason for that is, despite the low degree, these vertices exist in large amount. Even the existence of a vertex with degree close to  $t$  at time  $t$  may not be enough to freeze the diameter's growth. In [11] the authors have proven that the maximum degree of a modification of the BA-model is of order  $t$  at time  $t$ . However, the authors believe that this is not enough to obtain a diameter of order  $\log(\log(t))$ . To overcome these issues, one may modify the degree distribution by transposing mass to its tail, therefore increasing the maximum degree and decreasing the proportion of low degree vertices. Our model does this in a *non-artificial* way. By a local rule, introducing the edge-step functions (specifically the ones with polynomial decay), we are able to slow the growth of the total number of vertices by a large enough rate in order to break the diameter's growth. In this regard we have the following bounds:

**Theorem 4** (Lower bound on the diameter). *There exists a positive constant  $c_1$  such that*

$$(1.4) \quad \mathbb{P}\left(\text{diam}(G_t(\ell_M)) \leq c_1 \frac{\log(t)}{\log(\log(t))}\right) = o(1).$$

**Theorem 5** (Upper bound on the diameter). *If  $\gamma \in (0, 1)$ , then*

$$(1.5) \quad \mathbb{P}\left(\text{diam}(G_t(q_\gamma)) \geq 4 + \frac{12}{\gamma}\right) = o(1).$$

Theorem 4 shows that even though the “density”  $V(G_t(\ell_M))t^{-1}$  of vertices over time goes to 0 as  $t$  goes to infinity, the decay  $(\log t)^{-M}$  is hardly fast enough in order for one to observe a meaningful difference in behavior between the diameters of  $G_t(\ell_M)$  and of the original Barabási-Álbert graph, [3]. However, Theorem 5 states that the polynomial decay of  $q_\gamma$  is strong enough in order for one to observe a constant diameter, which implies a really small world phenomenon: a *minuscule world*!

See Table 1 for a summary of this brief discussion.

**1.4. Main technical ideas.** The proof of Theorem 3 and Corollary 3.12 regarding the existence of large complete subgraphs and lower bounds for the maximum degree comes from previous known results about most traditional models. The proofs are extended to the case of edge-step functions via a grand coupling which allows one to generate all the

Edge-step function	Power-Law exponent	Diameter a.a.s
$f(t) = p$	$2 + \frac{p}{2-p}$	$O(\log(t))$
$f(t) = 1/\log(t)$	2	$\geq \frac{\log(t)}{\log(\log(t))}$
$f(t) = 1/t^\gamma$	$2 - \gamma$	$O(1)$
$f(t) = 1/t$	<i>does not obey</i>	$O(1)$

TABLE 1. Comparison table among the different values for  $f(t)$ .

random graphs with different functions from the same source. This technique has proved very fruitful, since it also allows the conversion of bounds on the diameter of  $G_t(f)$ , for some  $f$ , to bounds on the diameter of  $G_t(g)$  by simply analyzing whether  $f$  is greater than  $g$  or not.

Some proofs require lower and upper bound estimates on the degree of the vertices. In this direction, we stress out our proof of Proposition 3.6 which assures an upper bound on later vertices' degree. It relies on a combination of Azuma's inequality (Theorem B.1) and Freedman's inequality (Theorem B.2). The main idea is a bootstrap argument in which first an upper bound for the degree is obtained via the traditional use of Azuma's inequality and then used on subsequent application of Freedman's inequality, this is the bootstrap phase. This phase may be iterated to obtain better and better upper bounds.

The proof of Theorem 4 is an application of the Payley-Zygmund inequality (or the second-moment method depending on personal name preferences). We show that the expectation of the random variable that counts the number of *isolated paths* of size  $C_1 \log(t)/\log(\log(t))$  goes to infinity as  $t \rightarrow \infty$ . By showing that the second moment of this random variable is very close to its first moment squared we prove that, with high probability,  $G_t(\ell_M)$  contains a large number of these paths. Finally, the existence of such a path will provide a lower bound for the diameter of  $G_t(\ell_M)$ .

The proof of Theorem 5 that assures a constant upper bound for  $\text{diam}(G_t(q_\gamma))$  makes use of special structures in  $G_t(q_\gamma)$ . We identify *w.h.p* the existence of a vertex with degree  $2t - t^{1-\delta}$  in  $G_t(q_\gamma)$ . This special vertex attracts all vertices with degree at least  $c_1 \log(t)$  giving origin to a subgraph of diameter constant and total degree close to  $t$ . Finally, controlling the degree of the remaining vertices and of the new vertices which are eventually added by the process we are able to give constant upper bounds on the their distance from the vertices with degree at least  $c_1 \log(t)$ .

**1.5. Organization.** In Section 2 we construct the grand coupling between the random graphs  $(G_t(f))_{t \geq 1}$  for every edge-step function  $f$ . From this grand coupling we prove the

existence of large cliques and some straightforward bounds for the diameter. In Section 3 we prove several technical estimates for the vertices' degree. Section 4 is devoted to the proof of a lower bound on the diameter of  $G_t(\ell_m)$ , whereas in Section 5 we prove a constant upper bound for the diameter of  $G_t(q_\gamma)$ . We leave to Appendix A the proofs of the power-law distribution and to Appendix B the statement of useful martingale concentration inequalities.

**1.6. Notation.** We let  $V(G_t(f))$  and  $E(G_t(f))$  denote the set of vertices and edges of  $G_t(f)$ , respectively. When the function  $f$  is clear from the context, we will denote the above sets simply by  $V_t$  and  $E_t$ . Given a vertex  $v \in V_t$ , we will denote by  $d_t(v)$  its degree in  $G_t(f)$ . We will also denote by  $\Delta d_t(v)$  the *increment* of the discrete function  $d_t(v)$  between times  $t$  and  $t + 1$ , that is,

$$\Delta d_t(v) = d_{t+1}(v) - d_t(v).$$

Given two sets  $A, B \subseteq V_t$ , we let  $\{A \leftrightarrow B\}$  denote the event where there exists an edge connecting a vertex from  $A$  to a vertex from  $B$ . We denote the complement of this event by  $\{A \nleftrightarrow B\}$ . We let  $\text{dist}(A, B)$  denote the graph distance between  $A$  and  $B$ , i.e. the minimum number of edges that a path that connects  $A$  to  $B$  must have. When one of these subsets consists of a single vertex, i.e.  $A = \{v\}$ , we drop the brackets from the definition and use  $\{v \leftrightarrow B\}$  and  $\text{dist}(v, B)$ , respectively.

Regarding constants, we let  $C_1, C_2, \dots$  and  $c, c_1, c_2, \dots$  be positive real numbers that do not depend on  $t$  whose values may vary in different parts of the paper. The dependence on other parameters will be highlighted throughout the text.

Since our model is inductive, we use the notation  $\mathcal{F}_t$  to denote the  $\sigma$ -algebra generated by all the random choices made up to time  $t$ . Then, we have the natural filtration  $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \dots$  associated to the process.

## 2. GRAND COUPLING

In this section we introduce a stochastic process  $(\mathcal{G}_t)_{t \geq 1}$  that provides a grand coupling between the random graphs  $(G_t(f))_{t \geq 1}$  for every function  $f : \mathbb{N} \rightarrow [0, 1]$ .

The process  $(\mathcal{G}_t)_{t \geq 1}$  is essentially a realization of the Barabási-Albert random tree where each vertex has two labels: an earlier vertex chosen according to the preferential attachment rule and an independent uniform random variable. The label consisting in the earlier vertex can be seen as a “ghost directed edge”, we later use these random labels to collapse subsets of vertices into a single vertex in order to obtain a graph with the same distribution as  $G_t(f)$  for any prescribed function  $f : \mathbb{N} \rightarrow [0, 1]$ .

We begin our process with a graph  $\tilde{G}_0$  consisting as usual in a single vertex and a single loop connecting said vertex to itself. We then inductively construct the labeled graph  $\tilde{G}_{t+1}$  from  $\tilde{G}_t$  in the following way:

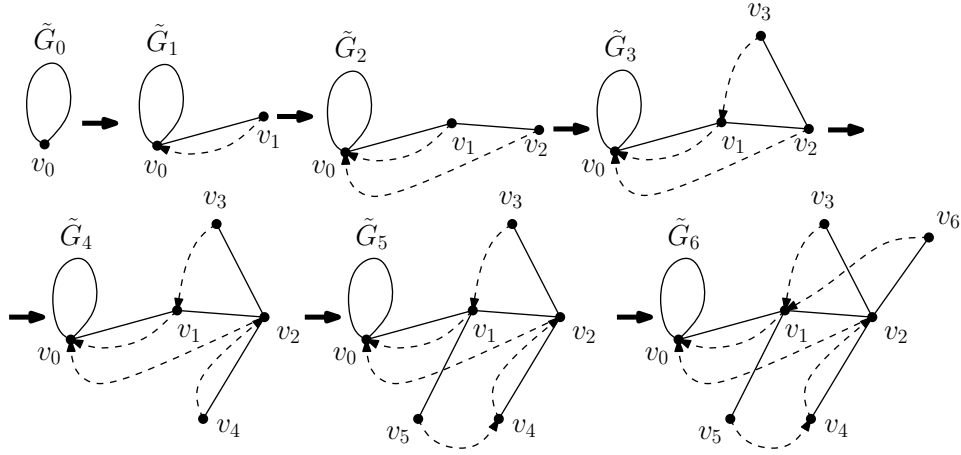


FIGURE 1. A sample of the process  $(\tilde{G}_t)_{t \geq 1}$  up to time 6. The dashed lines indicate the label  $\xi(v_j)$  taken by each vertex  $v_j$ .

- (i) We add to  $\tilde{G}_t$  a vertex  $v_{t+1}$ ;
- (ii) We tag  $v_{t+1}$  with a random label  $\xi(v_{t+1})$  chosen from the set  $V(\tilde{G}_t)$  with the preferential attachment rule, that is, with probability of choosing  $u \in V(\tilde{G}_t)$  proportional to the degree of  $u$  in  $\tilde{G}_t$ ;
- (iii) Independently from the step above, we add an edge  $\{e(v_{t+1}), v_{t+1}\}$  to  $E(\tilde{G}_t)$  where  $e(v_{t+1}) \in V(\tilde{G}_t)$  is also randomly chosen according to the preferential attachment rule.

We then obtain  $\mathcal{G}_t$  from  $\tilde{G}_t$  by tagging each vertex  $v_j \in V(\tilde{G}_t)$  with a second label consisting in an independent random variable  $U_j$  with uniform distribution on the interval  $[0, 1]$ . We now show how one can use the labeled graph  $\mathcal{G}_t$  to construct the distributions of  $(G_t(f))_{t \geq 1}$  for every  $f : \mathbb{N} \rightarrow [0, 1]$ . Let us fix such a function  $f$ . Given  $v_j \in V(\tilde{G}_t)$ , we compare  $U_j$  to  $f(j)$ . If  $U_j \leq f(j)$ , we do nothing. Otherwise, we collapse  $v_j$  onto its label  $\xi(v_j)$ , that is, we consider the set  $\{v_j, \xi(v_j)\}$  to be a single vertex with the same labels as  $\xi(v_j)$ . We then update the label of all vertices  $v$  such that  $\xi(v) = v_j$  to  $\{v_j, \xi(v_j)\}$ . This procedure is associative in the sense that the order of the vertices on which we perform this operation does not affect the final resulting graph, as long as we perform it for all the vertices of  $\tilde{G}_t$ . We claim that the resulting graph (after removing all remaining labels) has the same distribution as  $G_t(f)$ . To see the veracity of the above claim, one first notes that the associativity of the collapsing operation and the independence of the sequence  $(U_j)_{j \geq 1}$  from  $(\tilde{G}_t)_{t \geq 1}$  implies that we can glue together the vertex  $v_{t+1}$  to  $\xi(v_{t+1})$  whenever  $U_{t+1} > f(t+1)$  right after we complete step (iii) of the above construction by induction. The resulting graph has either a new vertex  $v_{t+1}$  with an edge  $\{e(v_{t+1}), v_{t+1}\}$  or an edge  $\{e(v_{t+1}), \xi(v_{t+1})\}$  with the exact same probability distribution as the  $(t+1)$ -th step in the construction of the graph  $(G_t(f))_{t \geq 1}$ . By induction, both random graphs have the same distribution.



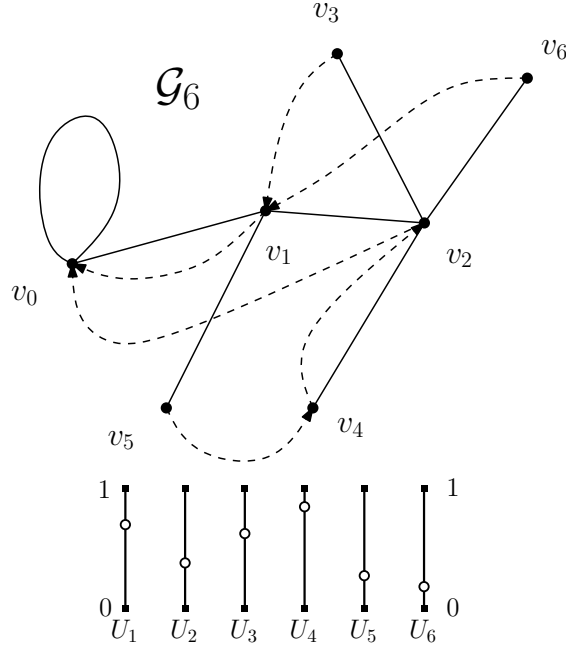


FIGURE 2. The labeled graph  $\mathcal{G}_6$ , constructed from  $\tilde{G}_6$  in Figure 1 by adding to each vertex  $v_j$  a second label consisting in an independent uniform random variable.

The above coupling has the following consequence for the comparison of the diameter of  $G_t(f)$  and  $G_t(g)$  when  $f \leq g$ :

**Corollary 2.1.** *Let  $f : \mathbb{N} \rightarrow [0, 1]$  and  $g : \mathbb{N} \rightarrow [0, 1]$  be such that  $f(n) \leq g(n)$  for every  $n \in \mathbb{N}$ . Then*

$$(2.2) \quad \text{diam}(G_t(f)) \preceq_{st} \text{diam}(G_t(g)),$$

where the symbol “ $\preceq_{st}$ ” denotes stochastic domination.

*Proof.* Follows immediately once one constructs both graphs using the labeled graph  $\mathcal{G}_t$  and notes that every path of length  $\text{diam}(G_t(f))$  in  $G_t(f)$  is also contained in  $G_t(g)$ .  $\square$

We note that  $|V(G_t(f))|$  is a sum of independent Bernoulli random variables with variable mean. Therefore, a concentration result about these types of variables will be an important tool here on out. We now state an elementary lemma in this direction that, together with the grand coupling, will prove general results about the graph  $G_t(f)$ .

**Lemma 1.** *Let  $(B_t)_{t \geq 1}$  be a sequence of i.i.d. Bernoulli random variables with parameter sequence given by  $(f(t))_{t \geq 1}$ . Define*

$$F(t) := \sum_{s=1}^t f(s).$$

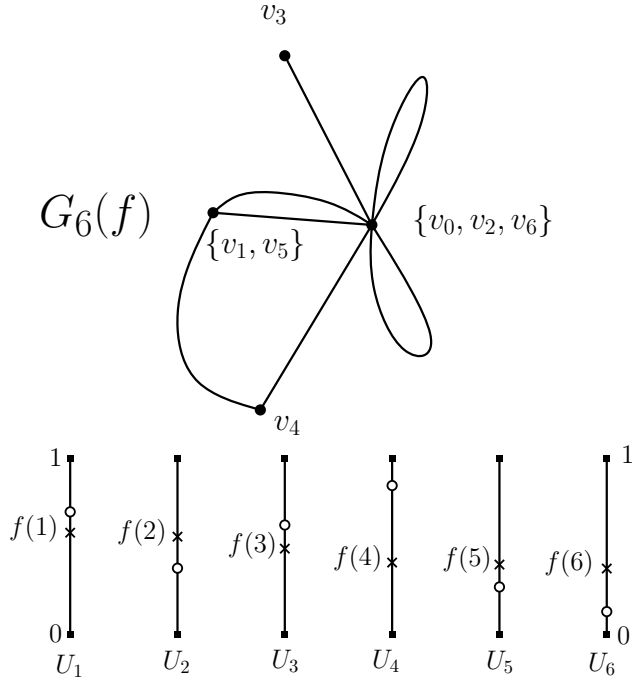


FIGURE 3. The figure shows how one can sample the distribution of  $G_6(f)$  using the labeled graph  $\mathcal{G}_6$ .

There exists a constant  $c > 0$  such that for every  $\varepsilon \in (0, 1)$ , we have

$$\mathbb{P} \left( (1 - \varepsilon)F_t \leq \sum_{s=1}^t B_s \leq (1 + \varepsilon)F_t \right) \geq 1 - 2 \exp(-c\varepsilon^2 F(t)).$$

*Proof.* The bound follows by a standard argument using the exponential Chebyshev's inequality.  $\square$

We obtain the following straightforward consequence of the above result and the grand coupling:

$$(2.3) \quad \mathbb{P}((1 - \varepsilon)F(t) \leq |V(G_t(f))| \leq (1 + \varepsilon)F(t)) \geq 1 - 2 \exp(-c\varepsilon^2 F(t)).$$

We note that

$$(2.4) \quad \frac{t}{(\log t)^M} \leq \mathbb{E}[|V(G_t(\ell_M))|] \leq t^{1-\varepsilon} + \sum_{k=t^{1-\varepsilon}}^t \frac{1}{(\log k)^M} \leq t^{1-\varepsilon} + \frac{t}{(1 - \varepsilon)^M (\log t)^M}$$

and, for  $\gamma \in (0, 1)$ ,

$$(2.5) \quad \int_1^t \frac{1}{x^\gamma} dx \leq \mathbb{E}[|V(G_t(q_\gamma))|] \leq 1 + \int_1^t \frac{1}{x^\gamma} dx \leq 1 + \frac{t^{1-\gamma}}{1-\gamma}.$$

In particular, (2.3) yields a stretched-exponential concentration inequality for  $|V(G_t(q_\gamma))|$  and  $|V(G_t(\ell_M))|$ .

We can then use these results together with Theorem 1 from [1] to prove Theorem 3

*Proof of Theorem 3.* Let  $p \in (0, 1)$  and  $\varepsilon \in (0, 1)$  be such that

$$\alpha = \alpha(p, \varepsilon) := \frac{1-p}{2-p}(1-\varepsilon) > \frac{1}{2} \left(1 - \frac{\delta}{2}\right).$$

From the proof of Theorem 1 of [1], we know that there exist a fixed integer  $m > 0$  and a small number  $\varepsilon' \in (0, \alpha)$  with the following property: if one divides the set of vertices born between times  $t^{\varepsilon'}$  and  $t^\alpha$  into disjoint subsets of  $m$  vertices born *consecutively*, then with high probability (at least 1 minus a polynomial function of  $t$ ) one can choose a vertex from each of these subsets in such a way that the subgraph induced by the set of chosen vertices is a complete subgraph of  $G_t(p)$ .

We assume that we constructed  $G_t(p)$  from  $\mathcal{G}_t$  and that  $t$  is large enough so that  $f(t^{\varepsilon'}) < p$ . Then one can sample  $G_t(f)$  using the same realization of the process  $\mathcal{G}_t$ . The complete subgraph of  $G_t(p)$  that we obtained will be preserved except for the vertices that will be collapsed when one constructs  $G_t(f)$  from  $\mathcal{G}_t$ . The probability that a vertex from  $G_t(p)$  that was born in time  $j$  survives the collapsing operation when one samples  $G_t(f)$  using  $\mathcal{G}_t$  is  $p^{-1}f(j)$ . Let  $(B_j)_{j \geq 1}$  be a sequence of independent Bernoulli random variables, each with probability of success given by  $p^{-1}f(j)$ . One can then construct in an elementary way a coupling where the size of the smallest complete subgraph of  $G_t(f)$  stochastically dominates the random variable

$$\sum_{j=t^{\varepsilon'}/m}^{t^\alpha/m} B_{jm}.$$

Since  $f$  is non-increasing, we know that

$$\begin{aligned} \mathbb{E} \left[ \sum_{j=t^{\varepsilon'}/m}^{t^\alpha/m} B_{jm} \right] &= \frac{1}{p} \sum_{j=t^{\varepsilon'}/m}^{t^\alpha/m} f(jm) \\ &\geq \frac{1}{mp} \sum_{j=t^{\varepsilon'}/m}^{t^\alpha/m} mf(jm) \\ &\geq \frac{1}{mp} \sum_{j=t^{\varepsilon'}}^{t^\alpha} f(j) \\ &\geq \frac{1}{mp} \left( F(t^\alpha) - F(t^{\varepsilon'}) \right). \end{aligned}$$

The fact that  $f(k) \geq k^{-1}$  then implies the existence of a constant  $c = c(m, \delta, \varepsilon') > 0$  such that the right hand side of the above inequality is uniformly (in  $t$ ) greater than  $cF(t^\alpha)$ . This

fact, together with the concentration result of Lemma 1, readily implies the existence of a complete subgraph of  $G_t(f)$  with the desired size asymptotically almost surely.  $\square$

Recall that  $\omega(G)$  denotes the size of the largest complete subgraph of the graph  $G$ . In particular, Theorem 3 implies that, with probability converging to 1 as  $t \rightarrow \infty$ , we have

$$(2.6) \quad \begin{aligned} \omega(G_t(\ell_M)) &\geq \frac{t^{(1-\delta)/2}}{(\log t)^M} \\ \omega(G_t(q_\gamma)) &\geq t^{(1-\delta)(1-\gamma)/2} \end{aligned}$$

for every  $\delta \in (0, 1)$ .

### 3. TECHNICAL ESTIMATES FOR THE DEGREE

This section is devoted to obtaining useful upper bounds for the degree of later vertices added by the process  $(G_t(f))_{t \geq 1}$  and to state a large enough lower bound for the degree of early vertices. The results stated and proved here are of a technical nature and may be skipped in a first reading.

**3.1. Upper bound.** We prove an upper bound for the degree of late vertices which holds independently of the edge-step function  $f$ . We start with the following result, which will be improved later:

**Lemma 2.** *Let  $v$  be a vertex added by the process  $(G_{2t}(f))_{t \geq 1}$  at time  $t$  and let  $M \in \mathbb{N}$ . We then have*

$$\mathbb{P}\left(d_{2t}(v) \geq 2\sqrt{8Mt \log(t)}\right) \leq t^{-M}.$$

*Proof.* For  $s \geq t$ , we have

$$(3.1) \quad \begin{aligned} \mathbb{E}[\Delta d_s(v) | \mathcal{F}_s] &= 1 \cdot \frac{d_s(v)}{2s} + 1 \cdot 2(1 - f(s+1)) \frac{d_s(v)}{2s} \left(1 - \frac{d_s(v)}{2s}\right) + 2 \cdot (1 - f(s+1)) \frac{d_s^2(v)}{4s^2} \\ &= \left(1 - \frac{f(s+1)}{2}\right) \frac{d_s(v)}{s}. \end{aligned}$$

Define the following random process, for  $s \geq t$ ,

$$(3.2) \quad M_s := \frac{d_s(v)}{\prod_{r=t}^{s-1} \left(1 + \frac{1}{r}\right)}$$

and observe that (3.1) implies that  $(M_s)_{s \geq t}$  is a supermartingale with  $M_t \equiv 1$ . Moreover, since we add at most a single vertex and a single edge at each step,  $|\Delta M_s|$  satisfies

$$(3.3) \quad |\Delta M_s| = \left| \frac{d_{s+1}(v) - \left(1 + \frac{1}{s}\right) d_s(v)}{\prod_{r=t}^s \left(1 + \frac{1}{r}\right)} \right| \leq \frac{2t}{s}$$

for all  $s \geq t$ . The above upper bound gives us that

$$\sum_{s=t}^{2t} |\Delta M_s|^2 \leq 4t^2 \sum_{s=t}^{2t} s^{-2} \leq 4t.$$

Now, applying Azuma's inequality (Theorem B.1) on  $M_{2t}$  and using the above upper bound we obtain

$$(3.4) \quad \mathbb{P}(d_{2t}(v) > 2(1 + \lambda)) \leq \mathbb{P}(M_{2t} - 1 > \lambda) \leq \exp \left\{ -\frac{\lambda^2}{8t} \right\}.$$

Choosing  $\lambda = \sqrt{8Mt \log(t)}$  we obtain

$$\mathbb{P}\left(d_t(v) > 2\sqrt{8Mt \log(t)}\right) \leq t^{-M},$$

which is the desired result.  $\square$

The above upper bound can be greatly improved by a bootstrap argument via Freedman's inequality (Theorem B.2), since Azuma's inequality requires deterministic upper bounds on the (super,sub)martingale's increments. The proof of this refined bound requires an upper bound on the square of the increments of the supermartingale defined on the above Lemma *conditioned on the whole past of the process*. With this in mind, we prove the following lemma:

**Lemma 3.** *Fix  $v \in V(G_t(f))$ . Then, for all  $s \geq t + 1$ , there exists a universal positive constant  $c_1$  such that*

$$\mathbb{E}[(\Delta M_s)^2 | \mathcal{F}_s] \leq c_1 \frac{t^2 d_s(v)}{s^3}.$$

*Proof.* The proof is straightforward

$$(3.5) \quad \mathbb{E}[(\Delta M_s)^2 | \mathcal{F}_s] = \mathbb{E} \left[ \frac{\left( \Delta d_s(v) - \frac{d_s(v)}{s} \right)^2}{\prod_{r=t}^s \left( 1 + \frac{1}{r} \right)^2} \middle| \mathcal{F}_s \right] \leq C \frac{d_s(v)}{s \cdot \frac{s^2}{t^2}},$$

where we expanded the squared term in the middle part of the above equation and used (3.1) with the fact that  $\Delta d_s(v) \leq 2 \cdot \mathbb{1}\{\Delta d_s(v) \geq 1\}$ .  $\square$

We can now state the main result of this subsection.

**Proposition 3.6.** *Let  $v$  be a vertex added by the process  $(G_t(f))_{t \geq 1}$  at time  $t$ . Then, for all  $N, M \in \mathbb{N}$ , there exists  $c = c(N, M)$  such that*

$$\mathbb{P}\left(d_{2t}(v) \geq ct^{\frac{1}{2^N}} \log^N(t)\right) \leq \frac{N}{t^M}.$$

*Proof.* We use Freedman's Inequality (Theorem B.2) and induction on  $N$ . The case  $N = 1$  is assured by Lemma 2. Therefore, we assume the proposition is proven for  $N - 1$ , meaning that there exists  $c(N - 1, M)$  such that

$$\mathbb{P} \left( d_{2t}(v) > c(N - 1, M) t^{\frac{1}{2^{N-1}}} \log^{N-1}(t) \right) \leq \frac{N - 1}{t^M}.$$

Now let  $B_{N-1,t}$  denote the following “bad” event

$$B_{N-1,t} := \left\{ d_{2t}(v) > c(N - 1, M) t^{\frac{1}{2^{N-1}}} \log^{N-1}(t) \right\},$$

and define  $X_s$  as

$$X_s := \sum_{r=t}^s \mathbb{E} \left[ (M_{r+1} - M_r)^2 | \mathcal{F}_r \right].$$

Combining the fact that  $d_t(v)$  is non-decreasing in  $t$  with Lemma 3 we are able to derive the following upper bound for  $X_s$

$$(3.7) \quad X_s \leq \sum_{r=t}^s c_2 \frac{t^2 d_r(v)}{r^3} \leq c_2 t^2 d_{2t}(v) \sum_{r=t}^s r^{-3} \leq c_3 d_{2t}(v)$$

where  $c_3$  is a universal positive constant. Moreover, the above upper bound and the definition of  $B_{N-1,t}$  gives us that, in the event  $B_{N-1,t}^c$ ,

$$\max_{t \leq s \leq 2t} X_s \leq c'(N - 1, M) t^{\frac{1}{2^{N-1}}} \log^{N-1}(t).$$

So, to simplify our writing, let  $\sigma^2$  be

$$\sigma^2 := c'(N - 1, M) t^{\frac{1}{2^{N-1}}} \log^{N-1}(t).$$

Hence, by Freedman's Inequality (Theorem B.2)

$$(3.8) \quad \mathbb{P} \left( M_{2t} - M_t > \lambda, \max_{t \leq s \leq 2t} X_s \leq \sigma^2 \right) \leq \exp \left( -\frac{\lambda^2}{8\lambda + 2\sigma^2} \right).$$

Choosing  $\lambda = c(N, M) \sigma \sqrt{\log(t)}$ , for a suitable constant  $c(N, M)$ , we obtain,

$$(3.9) \quad \mathbb{P} \left( M_{2t} - M_t > \lambda, \max_{t \leq s \leq 2t} X_s \leq \sigma^2 \right) \leq t^{-M}.$$

Finally, we have

$$(3.10) \quad \begin{aligned} \mathbb{P} \left( d_t(v) \geq c(N, M) t^{\frac{1}{2^N}} \log^N(t) \right) &\leq \mathbb{P} (M_{2t} - M_t > \lambda) \\ &\leq \mathbb{P} \left( M_{2t} - 1 > \lambda, \max_{t \leq s \leq 2t} X_s \leq \sigma^2 \right) + \mathbb{P} (B_{N-1}) \\ &\leq \frac{1}{t^M} + \frac{N - 1}{t^M} \end{aligned}$$

which proves the result. □

**3.2. Lower bound.** We start this section by stating a Corollary of [1] which assures the existence of vertices with very high degree on the random graph model with  $f(t) \equiv p$  for any value of  $p$ . According to the coupling introduced at Section 2, the result of this Corollary extends to the cases in which  $f$  is a function tending to zero as  $t$  goes to infinity. To that end, we use the following corollary of Theorem 2 of [1]:

**Corollary 3.11** (Lower bound for the degree, [1]). *Given  $p \in (0, 1)$ , let  $d_{\max}(G_t(p))$  be the largest degree among all the vertices of the random graph  $G_t(p)$ . Then for every  $\varepsilon > 0$ , we have*

$$\mathbb{P} \left( d_{\max}(G_t(p)) < t^{(1-\varepsilon)(1-p/2)} \right) \leq t^{-2}.$$

This bound then implies the following result

**Corollary 3.12.** *Let  $f$  be an edge-step function such that  $f(t) \rightarrow 0$  as  $t$  goes to infinity. Then, for any fixed  $\varepsilon > 0$  we have*

$$\mathbb{P} \left( d_{\max}(G_t(f)) < t^{1-\varepsilon} \right) \leq t^{-2}.$$

*Proof.* Corollary 3.11 guarantees *w.h.p* the existence of a vertex whose degree is at least  $t^{(1-\varepsilon')(1-p/2)}$  in  $G_t(p)$ , for any  $\varepsilon'$  and  $p \in (0, 1)$ . By letting these parameters be sufficiently small, we can assure *w.h.p* the existence of a vertex whose degree is at least  $t^{1-\varepsilon}$  in  $G_t(p)$ . Assuming that both  $G_t(p)$  and  $G_t(f)$  were generating using the coupling  $\mathcal{G}_t$ , one sees that there also exists *w.h.p.* in  $G_t(f)$  a vertex with degree at least  $t^{1-\varepsilon}$ .  $\square$

Throughout the rest of this subsection we will work on the process  $(G_t(q_\gamma))_{t \geq 1}$ . We will prove a proposition which assures a very intuitive fact: the first vertices added by the process should have large degree at time  $t$  when  $t$  increases. The proof requires upper bounds on the probability mass function of  $d_s(v)$  and may be too involved due to loops which eventually may be added to  $v$ . To avoid dealing with them, we couple the process  $(d_s(v))_{s \geq s_0}$ , starting at some specific time  $s_0$ , to a positive non-decreasing process  $(\tilde{d}_s)_{s \geq s_0}$  whose increments may be 0 or 1 only. We do this as follows

- when  $\tilde{d}_s = d_s(v) = j$ , put  $\tilde{d}_{s+1} = j + 1$  whenever  $d_{s+1}(v)$  jumps to  $j + 1$  or  $j + 2$ .
- otherwise, move  $\tilde{d}_{s+1}$  one step forward independently from  $d_{s+1}(v)$  with probability

$$(3.13) \quad \mathbb{P} \left( \tilde{d}_{s+1} = j + 1 \mid \tilde{d}_s = j, d_s(v) \neq j \right) = \left( 1 - \frac{f(s+1)}{2} \right) \frac{j}{s} - \frac{(1 - f(s+1))j^2}{4s^2}.$$

We will let both processes start from  $d_{s_0}(v)$ . Therefore, it follows from this above coupling that  $\tilde{d}_s \leq d_s(v)$  for all times  $s \geq s_0$ . Moreover, the transition probability of  $\tilde{d}_s$  is given by the right hand side of (3.13) whenever  $\tilde{d}_s = j$ .

For a fixed number  $\zeta > 0$ , let  $L_{t^{1/3}} = L_{t^{1/3}}(\zeta)$  be the following random set of vertices

$$(3.14) \quad L_{t^{1/3}} := \{v \in V(G_{t^{1/3}}(q_\gamma)), d_{t^{1/3}}(v) < \zeta \log(t)\}.$$

That is,  $L_{t^{1/3}}$  is the set of vertices added by the process  $(G_t(q_\gamma))_{t \geq 1}$  before time  $t^{1/3}$  whose degrees are less than some specific positive number times  $\log(t)$ . We are going to prove that *w.h.p* all vertices in  $L_{t^{1/3}}$  have degree at least  $\zeta \log(t)$  at time  $t$ , for a suitable choice of  $\zeta$ .

**Proposition 3.15** (Older vertices have large degree). *Consider the process  $(G_t(q_\gamma))_{t \geq 1}$ . Then, for every fixed constant  $C_1$ , there exist positive constants  $C_2$  and  $C_3$  such that*

$$\mathbb{P} \left( \bigcup_{v \in L_{t^{1/3}}} \{d_t(v) \leq \zeta \log(t)\} \right) \leq C_2 \frac{\zeta^2 \log^2(t)}{t^{1/3}}$$

*Proof.* Recall that we denote the *vertex born at time  $i$*  by  $v_i$ . We bound the *c.d.f.* of  $d_t(v_i)$ , where  $i \leq t^{1/3}$  and conclude via an union bound. To do this, we begin recalling the transition probabilities of  $d_s(v)$  implied by (3.1) which in turn imply, for all  $s_0 < t$  and  $j \geq 1$ , the following bound:

$$\begin{aligned} \mathbb{P}(d_t(v) = j | d_{s_0}(v) = j) &= \prod_{r=s_0}^{t-1} \left( 1 - \frac{(1 - (r+1)^{-\gamma}/2)j}{r} + \frac{(1 - (r+1)^{-\gamma}/2)j^2}{4r^2} \right) \\ (3.16) \quad &\leq \prod_{r=s_0}^{t-1} \left( 1 - \frac{j}{r} + \frac{j^2}{r^{1+\gamma}} \right) \\ &\leq \frac{s_0^j}{(t-1)^j} \exp \left\{ j^2 \sum_{r=s_0}^{t-1} r^{-1+\gamma} \right\}. \end{aligned}$$

The exponential term on the above upper bound will appear in what follows, so, to simplify our writing, we give it a special notation:

$$(3.17) \quad g(j, s_0, t) := \exp \left\{ j^2 \sum_{r=s_0}^{t-1} r^{-1+\gamma} \right\}.$$

We now proceed by coupling the random process  $(d_s(v_i))$  conditioned on  $d_{s_0}(v_i) = j$  to the process  $\tilde{d}_s$ , both starting from  $j$ . Since  $\tilde{d}_s \leq d_s(v_i)$ , it is enough to obtain a useful upper bound on the  $\tilde{d}_t$ 's *p.m.f.* We do this by considering all possible times  $\tilde{d}$  may jump until it reaches a certain value  $k \in \mathbb{Z}_+$ . So, for now on,  $k$  will be a positive integer smaller than  $\zeta \log(t)$  and  $A_{t_1, \dots, t_{k-j}}$  will denote the intersection of events defined below

$$(3.18) \quad A_{t_1, \dots, t_{k-j}} := \bigcap_{s \in (s_0, t] \setminus \{t_1, \dots, t_{k-j}\}} \left\{ \Delta \tilde{d}_{s-1} = 0 \right\} \bigcap_{i=1}^{k-1} \left\{ \Delta \tilde{d}_{t_i-1} = 1 \right\}.$$

We note that the probability of  $\tilde{d}_s$  staying put is identical to the probability of this same event for the process  $(d_s(v))_{s \geq s_0}$ . Therefore, using the upper bound (3.16) and the fact that



the process  $\tilde{d}_s$  is markovian (though not homogeneous in time), we obtain

$$(3.19) \quad \begin{aligned} \mathbb{P} \left( A_{t_1, \dots, t_{k-j}} \middle| \tilde{d}_{s_0} = j \right) &\leq \frac{s_0^j}{(t_1 - 2)^j} g(j, s_0, t_1 - 1) \frac{j}{2(t_1 - 1)} \frac{t_1^{j+1}}{(t_2 - 2)^{j+1}} g(j+1, t_1, t_2 - 1) \\ &\quad \times \frac{j+1}{2(t_2 - 1)} \frac{t_2^{j+2}}{(t_3 - 2)^{j+2}} g(j+2, t_2, t_3 - 1) \\ &\quad \times \dots \times \frac{k}{2(t_{k-j} - 1)} \frac{t_{k-j}^k}{(t - 1)^k} g(k, t_{k-j}, t). \end{aligned}$$

By the definition of the  $g$  functions, their product is bounded by

$$(3.20) \quad g(j, s_0, t_1 - 1) \dots g(k, t_{k-j}, t) = \exp \left\{ j^2 \sum_{r=s_0}^{t_1-1} r^{-1+\gamma} + \dots + k^2 \sum_{r=t_{k-j}}^{t-1} r^{-1+\gamma} \right\} \leq \exp \{ \gamma^{-1} k^2 s_0^{-\gamma} \}$$

which gives us

$$(3.21) \quad \mathbb{P} \left( A_{t_1, \dots, t_{k-j}} \middle| \tilde{d}_{s_0} = j \right) \leq \frac{s_0^j}{(t - 1)^k} k(k-1) \dots j \exp \{ \gamma^{-1} k^2 s_0^{-\gamma} \} \prod_{i=1}^{k-j} \frac{t_i^{j+i}}{(t_i - 2)^{j+i-1} (t_i - 1)}.$$

Since we are interested in bounding from above  $\mathbb{P}(\tilde{d}_t = k | \tilde{d}_{s_0} = j)$ , we sum the upper bound given by (3.21) over all possible choices for the sequence  $t_1, \dots, t_{k-j}$ , noticing that

$$(3.22) \quad \sum_{s_0 \leq t_1 < \dots < t_{k-j} \leq t} \prod_{i=1}^{k-j} \frac{t_i^{j+i}}{(t_i - 2)^{j+i-1} (t_i - 1)} \leq \frac{1}{(k-j)!} \sum_{s_0 \leq t_1, \dots, t_{k-j} \leq t} \prod_{i=1}^{k-j} \left( 1 + \frac{2}{t_i - 2} \right)^{j+i}.$$

Moreover,

$$\sum_{r=s_0}^t \left( 1 + \frac{2}{r-2} \right)^{j+i} \leq t \left( 1 + \frac{2}{s_0-2} \right)^{j+i}$$

which leads us to

$$(3.23) \quad \begin{aligned} \sum_{s_0 \leq t_1 < \dots < t_k \leq t} \prod_{i=1}^{k-j} \frac{t_i^{j+1}}{(t_i - 2)^j (t_i - 1)} &\leq \frac{t^{k-j}}{(k-j)!} \prod_{i=1}^{k-j} \left( 1 + \frac{2}{s_0 - 2} \right)^{j+i} \\ &\leq \frac{t^{k-j}}{(k-j)!} \exp \{ 2k^2 s_0^{-1} \}. \end{aligned}$$

This yields the upper bound below

$$(3.24) \quad \mathbb{P} \left( \tilde{d}_t = k \middle| \tilde{d}_{s_0} = j \right) \leq \frac{s_0^j}{t^j} \binom{k}{j} \exp \{ 3\gamma^{-1} k^2 s_0^{-\gamma} \}.$$

Summing over  $k \leq \zeta \log(t)$  and recalling that  $\tilde{d}_s \leq d_s(v_i)$  for all  $s$ , we have

$$(3.25) \quad \mathbb{P}(d_t(v_i) \leq \zeta \log(t) | d_{s_0}(v_i) = j) \leq C_2 \frac{s_0^j}{t^j} \binom{\zeta \log(t) + 1}{j+1} \exp \{C_3 \log^2(t) s_0^{-\gamma}\}.$$

Finally, recalling from the coupling that the vertex  $v_i$  exists if, and only if, its uniform  $U_i$  is less than  $f(i)$ , we have

$$\begin{aligned} \mathbb{P}(d_t(v_i) \leq \zeta \log(t) | U_i \leq f(i)) &= \sum_{j=1}^{\zeta \log(t)} \mathbb{P}(d_t(v_i) \leq \zeta \log(t), d_{s_0}(v_i) = j | U_i \leq f(i)) \\ &\leq C_2 \exp \{C_3 \log^2(t) s_0^{-\gamma}\} \sum_{j=1}^{\zeta \log(t)} \frac{s_0^j}{t^j} \binom{\zeta \log(t) + 1}{j+1} \\ &\leq C_4 \frac{s_0 \zeta^2 \log^2(t)}{t} \exp \{C_3 \log^2(t) s_0^{-\gamma}\}. \end{aligned}$$

Setting  $s_0 = t^{1/3}$  and combining the above upper bound with a union bound on the vertices in  $L_{t^{1/3}}$ , we obtain the desired result since  $|L_{t^{1/3}}| \leq t^{1/3}$ .  $\square$

#### 4. A $\log / \log \log$ LOWER BOUND FOR DIAMETER OF $G_t(\ell_M)$ : PROOF OF THEOREM 4

We begin by observing that together with Corollary 2.1, Theorem 4 implies that if  $f : \mathbb{N} \rightarrow [0, 1]$  has a slower decay than  $\ell_M$ , then  $\text{diam}(G_t(f))$  is also larger than  $C_1 \log(t) / \log(\log(t))$  with high probability. Therefore this section is also an alternate proof of the lower bound for the BA-model, [3].

We start by defining precisely what we mean by an isolated path.

**Definition 1** (Isolated path). *Let  $l$  be a positive integer. Let  $\vec{t} = (t_1, \dots, t_l)$  be a vector of distinct positive integers. We say that this vector corresponds to an **isolated path**  $\{v_{t_1}, \dots, v_{t_l}\}$  in  $G_t(\ell_M)$  if and only if:*

- $t_l \leq t$ ;
- $t_i < t_j$  whenever  $1 \leq i < j \leq l$ ;
- during each time  $t_i$ ,  $i = 1, \dots, l$ , the inductive construction of  $(G_t(\ell_M))_{t \geq 1}$  performed an vertex-step;
- for every integer  $k \leq l$ , the subgraph induced by the vertices  $\{v_{t_i}\}_{1 \leq i \leq k}$  is connected in  $G_{t_k}(\ell_M)$ ;
- for  $i = 1, \dots, l-1$ , the degree of  $v_{t_i}$  in  $G_t(\ell_M)$  is 2. The degree of  $v_{t_l}$  in  $G_t(\ell_M)$  is 1.

In other words, an isolated path  $\{v_{t_i}\}_{1 \leq i \leq l}$  is a path where each vertex  $v_{t_i}$ , for  $i = 2, \dots, l$ , is born at time  $t_i$  and makes its first connection to its predecessor  $v_{t_{i-1}}$ . Other than that, no other vertex or edge gets attached to  $\{v_{t_i}\}_{1 \leq i \leq l}$ . We will denote  $\{v_{t_i}\}_{1 \leq i \leq l}$  by  $v_{\vec{t}}$ .

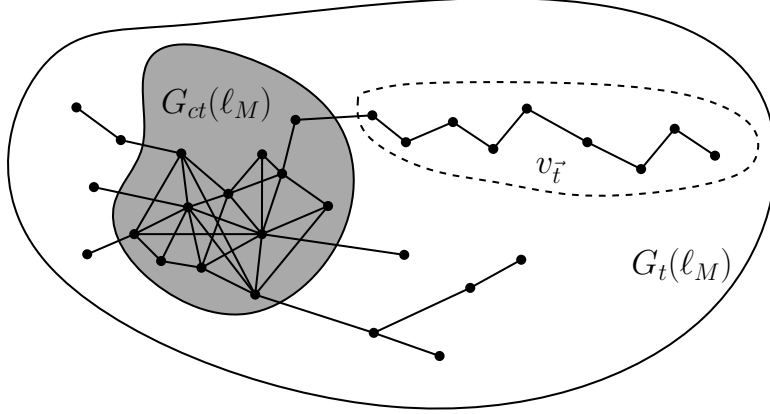


FIGURE 4. An example of an isolated path formed by vertices born between times  $ct$  and  $t$ .

Let  $\mathcal{S}_l(t)$  be the set containing all isolated paths of size  $l$  in  $G_t(l_M)$ . Given  $c \in (0, 1)$ , denote by  $N_{ct,t}^l$  the number of isolated paths whose vertices were created at times between  $ct$  and  $t$ . Our first order of business is to obtain lower bounds for  $\mathbb{E}[N_{ct,t}^l]$ :

**Lemma 4.** *For any  $0 < c < 1$  and any integer  $l \leq ct$ , the following lower bound to  $\mathbb{E}[N_{ct,t}^l]$  holds:*

$$(4.1) \quad \mathbb{E}[N_{ct,t}^l] \geq \binom{(1-c)t}{l} \frac{1}{(2t)^{l-1} \log^{Ml}(t)} \left(1 - \frac{2l}{ct}\right)^t.$$

*Proof.* The random variable  $N_{ct,t}^l$  can be written as

$$(4.2) \quad N_{ct,t}^l = \sum_{\substack{t_1 < t_2 < \dots < t_l \\ t_i \in [ct, t]}} \mathbb{1}_{\{v_{\vec{t}} \in \mathcal{S}_l(t)\}}.$$

Then, its expected value is

$$\mathbb{E}[N_{ct,t}^l] = \sum_{\substack{t_1 < t_2 < \dots < t_l \\ t_i \in [ct, t]}} \mathbb{P}(v_{\vec{t}} \in \mathcal{S}_l(t)).$$

So it suffices to obtain a proper lower bound to  $\mathbb{P}(v_{\vec{t}} \in \mathcal{S}_l(t))$ . Given  $\vec{t} = (t_1, \dots, t_l)$  such that  $ct \leq t_1 < t_2 < \dots < t_l \leq t$ , we claim that

$$\mathbb{P}(v_{\vec{t}} \in \mathcal{S}_l(t)) \geq \frac{1}{(2t)^{l-1} \log^{Ml}(t)} \left(1 - \frac{2l}{ct}\right)^t.$$

To see why the above claim is true, observe that in order for  $v_{\vec{t}}$  to be in  $\mathcal{S}_l(t)$ , we need to assure that  $l$  vertices are born exactly at times  $t_1, \dots, t_l$  (which happens with probability greater than  $\log^{-Ml}(t)$ ), that  $v_{t_i}$  connects to  $v_{t_{i-1}}$  for every  $i = 2, \dots, l$  (which happens with

probability greater than  $(2t)^{-(l-1)}$ , and that no other vertex or edge connects to  $v_{\vec{t}}$  until time  $t$  (which happens with probability greater than  $(1 - \frac{2l}{ct})^t$ ).

Finally, by counting the number of possible ways to choose  $t_1 < t_2 < \dots < t_l$  so that  $t_i \in [ct, t]$  for  $1 \leq i \leq l$ , we obtain the desired lower bound for  $\mathbb{E}[N_{ct,t}^l]$ .  $\square$

**Lemma 5.** *There exist positive constants  $c_1, c_2$  and  $c_3$  such that, for  $l = c_2 \frac{\log(t)}{\log(\log(t))}$ ,*

$$\mathbb{E}[N_{c_1 t, t}^l] \geq c_3 t^{c_3}.$$

*Proof.* We will use the lower bound provided by Lemma 4 and substitute  $l$  by  $c_2 \frac{\log(t)}{\log(\log(t))}$ . Since  $l \ll t$ , Stirling's formula gives us

$$\binom{(1-c_1)t}{l} \geq c(1-c_1)^l t^l e^l l^{-l-\frac{1}{2}},$$

for some constant  $c > 0$ . Choosing  $c_1 = 1 - e^{-1}$ , we obtain

$$\binom{(1-c_1)t}{l} \geq ct^l l^{-l-\frac{1}{2}} \geq c \exp\{l \log(t) - 2l \log(l)\}.$$

Combining the above inequality with (4.2) gives us, for large enough  $t$  and a different constant  $c > 0$ ,

$$\mathbb{E}[N_{c_1 t, t}^l] \geq c \exp\{l \log(t) - 2l \log(l) - (l-1) \log(2t) - Ml \log(\log(t)) - 2c_1^{-1}l\}.$$

By choosing  $c_2 = c_2(M)$  appropriately and substituting  $l = c_2 \frac{\log(t)}{\log(\log(t))}$  we get

$$\mathbb{E}[N_{c_1 t, t}^l] \geq c_3 t^{c_3}$$

for some constant  $c_3 > 0$ . This finishes the proof of the lemma.  $\square$

Some new notation will be useful throughout the proof of Theorem 4:

**Definition 2** (Degree of an isolated path). *Given an isolated path  $v_{\vec{t}} = \{v_{t_j}\}_{1 \leq j \leq l}$ , we denote by  $d_r(v_{\vec{t}})$  the sum of the degrees of each of its vertices at time  $r$ , i.e. :*

$$d_r(v_{\vec{t}}) = \sum_{t_i \in \vec{t}} d_r(v_{t_i}),$$

where we assumed that  $d_r(v_{t_i}) = 0$  if  $t_i < l$ .

Note that if  $r > t_l$  and  $v_{\vec{t}}$  has size  $l$ , then  $d_r(v_{\vec{t}}) = 2l - 1$ . Furthermore,

$$(4.3) \quad \mathbb{P}(\Delta d_s(v_{\vec{t}}) > 0 \mid G_s(\ell_M)) = 1 - \left(1 - \frac{1}{2(\log(s+1))^M}\right) \frac{d_s(v_{\vec{t}})}{s}.$$

We now complete the proof of the main theorem of this subsection.

*Proof of Theorem 4.* Let  $c_1 > 0$  be the same constant  $c_1$  defined in Lemma 5, and let  $l = c_2 \frac{\log(t)}{\log(\log(t))}$  as before. Paley-Zigmond's inequality assures us that, for any  $0 \leq \theta \leq 1$ ,

$$(4.4) \quad \mathbb{P}(N_{c_1 t, t}^l > \theta \mathbb{E}[N_{c_1 t, t}^l]) \geq (1 - \theta)^2 \frac{(\mathbb{E}[N_{c_1 t, t}^l])^2}{\mathbb{E}[(N_{c_1 t, t}^l)^2]}.$$

If we are able to guarantee that

- (i)  $\mathbb{E}[N_{c_1 t, t}^l] \rightarrow \infty$ ;
- (ii)  $(\mathbb{E}[N_{c_1 t, t}^l])^2 = (1 - o(1))\mathbb{E}[(N_{c_1 t, t}^l)^2]$ ;

by choosing  $\theta = \theta(t)$  such that  $\theta(t)$  goes to zero slower than  $\mathbb{E}[N_{c_1 t, t}^l]$  goes to infinity, then we will have finished the proof of the theorem.

By Lemma 5, we know that item (i) is true. Therefore from now on we will focus on proving item (ii).

In order to create the isolated path  $v_{\vec{t}}$ , we use the following recipe:

- a vertex  $v_{t_1}$  must be created at time  $t_1$ ;
- between times  $t_1 + 1$  and  $t_2 - 1$  we allow no new connection to  $v_{t_1}$ ;
- at time  $t_2$  a new vertex  $v_{t_2}$  is created and makes its first connection to  $v_{t_1}$ ;
- we continue the process, creating a vertex  $v_{t_k}$  that makes its first connection  $v_{t_{k-1}}$ , and letting no new connection be made to  $\{v_{t_j}\}_{1 \leq j \leq k}$  between times  $t_k + 1$  and  $t_{k+1} - 1$  for every  $k = 2, \dots, l - 1$ ;
- finally, a vertex  $v_{t_l}$  is born at time  $t_l$  and we let no connection be made to  $\{v_{t_j}\}_{1 \leq j \leq l}$  between times  $t_l + 1$  and  $t$ .

This implies

$$(4.5) \quad \begin{aligned} \mathbb{P}(v_{\vec{t}} \in \mathcal{S}_l(t)) &= \frac{1}{(\log(t_1))^M} \prod_{r_1=t_1+1}^{t_2-1} \left(1 - \left(1 - \frac{1}{2(\log(r_1))^M}\right) \frac{1}{r_1 - 1}\right) \times \\ &\times \frac{1}{(\log(t_2))^M} \frac{1}{2(t_2 - 1)} \prod_{r_2=t_2+1}^{t_3-1} \left(1 - \left(1 - \frac{1}{2(\log(r_2))^M}\right) \frac{3}{r_2 - 1}\right) \times \\ &\times \dots \times \prod_{r_l=t_l+1}^t \left(1 - \left(1 - \frac{1}{2(\log(r_l))^M}\right) \frac{2l - 1}{r_l - 1}\right). \end{aligned}$$

Given two time vectors  $\vec{r}$  and  $\vec{t}$ , we note that  $\mathbb{P}(v_{\vec{t}}, v_{\vec{r}} \in \mathcal{S}_l(t))$  is only nonzero if  $\vec{r}$  and  $\vec{t}$  have either disjoint or identical sets of entries. Our focus now is on proving the following claim:

**Claim 1.** *For two isolated paths with disjoint time vectors, we have*

$$\mathbb{P}(v_{\vec{t}}, v_{\vec{r}} \in \mathcal{S}_l(t)) \leq (1 + o(1)) \mathbb{P}(v_{\vec{t}} \in \mathcal{S}_l(t)) \mathbb{P}(v_{\vec{r}} \in \mathcal{S}_l(t)).$$

To prove the above equation we will make a comparison between the terms  $\mathbb{P}(v_{\vec{t}}, v_{\vec{r}} \in \mathcal{S}_l(t))$  and  $\mathbb{P}(v_{\vec{t}} \in \mathcal{S}_l(t)) \mathbb{P}(v_{\vec{r}} \in \mathcal{S}_l(t))$ . We can write both these terms as products in the manner of (4.5). We can then compare the terms from these products associated to each time  $s \in [c_1 t, t]$ . There are two cases we must study.

*Case 1:  $s \in \vec{t}$  but  $s \notin \vec{r}$  ( $s \notin \vec{t}$  but  $s \in \vec{r}$ ).*

The product term related to time  $s$  in  $\mathbb{P}(v_{\vec{t}}, v_{\vec{r}} \in \mathcal{S}_l(t))$  is

$$(4.6) \quad \frac{1}{(\log(s))^M} \frac{1}{2(s-1)},$$

since a new vertex is created and then makes its first connection specifically to the latest vertex of  $\vec{t}$ . On the other hand, the term related to time  $s$  in  $\mathbb{P}(v_{\vec{t}} \in \mathcal{S}_l(t)) \mathbb{P}(v_{\vec{r}} \in \mathcal{S}_l(t))$  is

$$\frac{1}{(\log(s))^M} \frac{1}{2(s-1)} \left( 1 - \left( 1 - \frac{1}{2(\log(s))^M} \right) \frac{d_{s-1}(v_{\vec{r}})}{s-1} \right),$$

since the term related to  $s$  in the product form of  $\mathbb{P}(v_{\vec{t}} \in \mathcal{S}_l(t))$  continues to be equal to (4.6), but the related term in  $\mathbb{P}(v_{\vec{r}} \in \mathcal{S}_l(t))$  is

$$(4.7) \quad \left( 1 - \left( 1 - \frac{1}{2(\log(s))^M} \right) \frac{d_{s-1}(v_{\vec{r}})}{s-1} \right).$$

The above expression is the term that will appear regarding the time  $s$  in the fraction

$$(4.8) \quad \frac{\mathbb{P}(v_{\vec{t}} \in \mathcal{S}_l(t)) \mathbb{P}(v_{\vec{r}} \in \mathcal{S}_l(t))}{\mathbb{P}(v_{\vec{t}}, v_{\vec{r}} \in \mathcal{S}_l(t))}.$$

This case occurs  $2l$  times since the isolated paths are disjoint. Thus, recalling that  $s \in [c_1 t, t]$ ,  $l = c_2 \log(t) / \log(\log(t))$  and that the degree of each isolated path is at most  $2l - 1$ , we obtain that there exist constants  $c_4, c_5 > 0$  such that we can bound the product of all the terms of the form (4.7) from above by

$$\left( 1 - \frac{c_4}{t} \right)^{2l},$$

and from below by

$$\left( 1 - \frac{c_5 l}{t} \right)^{2l}.$$

Observe that both products go to 1 as  $t$  goes to infinity.

*Case 2:  $s \notin \vec{t}$  and  $s \notin \vec{r}$ .*

In  $\mathbb{P}(v_{\vec{r}} \in \mathcal{S}_l(t))$  as well as in  $\mathbb{P}(v_{\vec{t}} \in \mathcal{S}_l(t))$  we see terms of the form (4.7), since we must avoid the isolated paths in both events. But in  $\mathbb{P}(v_{\vec{t}}, v_{\vec{r}} \in \mathcal{S}_l(t))$  we observe

$$1 - \left( 1 - \frac{1}{2(\log(s))^M} \right) \frac{(d_{s-1}(v_{\vec{t}}) + d_{s-1}(v_{\vec{r}}))}{s-1},$$

since we must guarantee that neither isolated path receives a connection. Now, observe that for two real numbers  $a, b > 0$ , we have

$$\left(1 - \frac{a}{s}\right) \left(1 - \frac{b}{s}\right) = \left(1 - \frac{(a+b)}{s}\right) \left(1 + \frac{ab}{s^2(1 - o(1))}\right).$$

In the fraction (4.8), we have  $\Theta(t)$  terms of the form

$$\left(1 + \frac{d_{s-1}(v_{\vec{t}})d_{s-1}(v_{\vec{r}})}{s^2(1 - o(1))}\right),$$

with  $s \geq ct$ . But again, as in Case 1 their product goes to 1 as  $t \rightarrow \infty$ . This proves the claim.

Now, observe that

$$\begin{aligned} \mathbb{E} \left[ (N_{ct,t}^l)^2 \right] &= \mathbb{E} \left[ \left( \sum_{\vec{t}} \mathbb{1} \{v_{\vec{t}} \in \mathcal{S}_l(t)\} \right) \left( \sum_{\vec{r}} \mathbb{1} \{v_{\vec{r}} \in \mathcal{S}_l(t)\} \right) \right] \\ (4.9) \quad &= \sum_{\substack{\vec{t}, \vec{r} \\ \text{disjoint}}} \mathbb{P}(v_{\vec{t}}, v_{\vec{r}} \in \mathcal{S}_l(t)) + \mathbb{E} [N_{ct,t}^l], \end{aligned}$$

and that

$$\left( \mathbb{E} [N_{ct,t}^l] \right)^2 = \sum_{\substack{\vec{t}, \vec{r} \\ \text{disjoint}}} \mathbb{P}(v_{\vec{t}} \in \mathcal{S}_l(t)) \mathbb{P}(v_{\vec{r}} \in \mathcal{S}_l(t)) + \sum_{\substack{\vec{t}, \vec{r} \\ \vec{r} \cap \vec{t} \neq \emptyset}} \mathbb{P}(v_{\vec{t}} \in \mathcal{S}_l(t)) \mathbb{P}(v_{\vec{r}} \in \mathcal{S}_l(t)).$$

Therefore, using the equations above and Lemma 5, we obtain

$$\begin{aligned} \frac{\mathbb{E} \left[ (N_{ct,t}^l)^2 \right]}{\left( \mathbb{E} [N_{ct,t}^l] \right)^2} &= \frac{\sum_{\substack{\vec{t}, \vec{r} \\ \text{disjoint}}} \mathbb{P}(v_{\vec{t}}, v_{\vec{r}} \in \mathcal{S}_l(t))}{\left( \mathbb{E} [N_{ct,t}^l] \right)^2} + \frac{\mathbb{E} [N_{ct,t}^l]}{\left( \mathbb{E} [N_{ct,t}^l] \right)^2} \\ (4.10) \quad &\leq \frac{\sum_{\substack{\vec{t}, \vec{r} \\ \text{disjoint}}} \mathbb{P}(v_{\vec{t}}, v_{\vec{r}} \in \mathcal{S}_l(t))}{\sum_{\substack{\vec{t}, \vec{r} \\ \text{disjoint}}} \mathbb{P}(v_{\vec{t}} \in \mathcal{S}_l(t)) \mathbb{P}(v_{\vec{r}} \in \mathcal{S}_l(t))} + \frac{1}{\mathbb{E} [N_{ct,t}^l]} \\ &\leq \frac{\sum_{\substack{\vec{t}, \vec{r} \\ \text{disjoint}}} (1 + o(1)) \mathbb{P}(v_{\vec{t}} \in \mathcal{S}_l(t)) \mathbb{P}(v_{\vec{r}} \in \mathcal{S}_l(t))}{\sum_{\substack{\vec{t}, \vec{r} \\ \text{disjoint}}} \mathbb{P}(v_{\vec{t}} \in \mathcal{S}_l(t)) \mathbb{P}(v_{\vec{r}} \in \mathcal{S}_l(t))} + \frac{1}{\mathbb{E} [N_{ct,t}^l]} \\ &\leq 1 + o(1), \end{aligned}$$

which proves the desired result.  $\square$

## 5. CONSTANT UPPER BOUND FOR THE DIAMETER OF $G_t(q_\gamma)$ : PROOF OF THEOREM 5

In this Section we prove the main theorem regarding the diameter. The result guarantees that *w.h.p* the graphs generated by the process  $(G_t(q_\gamma))_t$ , for  $\gamma \in (0, 1)$  have diameter bounded from above by  $4 + 12/\gamma^{-1}$ . For organizational matters, we split the proof into several lemmas which are, by themselves, a constructive way to see why  $G_t(q_\gamma)$  must have such a short diameter.

We will use the term *strength* of a given set of vertices  $J \subseteq V_t$  to denote how the total degree of  $J$ ,  $d_t(J) = \sum_{v \in J} d_t(v)$  compares to the total degree of  $V_t$ ,  $d_t(V_t) = 2t$ . We say that  $J$  is *strong* when it has high degree when compared to  $V_t$ .

We begin by recalling Corollary 3.12 which assures the existence, *w.h.p*, of a vertex  $v^* \in V_t$  whose degree at time  $t$  is at least  $t^{1-\varepsilon}$  for  $\varepsilon$  which can be chosen as small as we want. We call  $v^*$  the *star*. The existence of the star will later imply the existence of an extremely strong subgraph  $H_t$  of  $G_t(q_\gamma)$  of  $V_t$  in the sense that  $d_t(H_t) = t(1 - o(1))$ . This subgraph will play an important role in the task of shortening the diameter of  $G_t(q_\gamma)$ , since the newer edges will have a probability bounded away from zero to connect themselves to  $H_t$  at each step. We call every connected subgraph of  $G_t(q_\gamma)$  whose degree is at least  $c \cdot t$ , for some positive constant  $c$ , an *attractive graph*. The theorem below assures us the existence *w.h.p* of at least one attractive graph with diameter 2 in  $G_t(q_\gamma)$ .

**Lemma 6** (The Attractive Graph with small diameter). *If  $\delta \in (0, 1)$  is small enough, then*

$$\mathbb{P}(\exists H_{2t} \subset G_{2t}(q_\gamma), \text{ connected with } d_{2t}(H_{2t}) \geq 2t - t^{1-\delta}) \geq 1 - 2t^{-2}.$$

*Proof.* As we said, by Corollary 3.12 there exists *w.h.p* a vertex  $v_* \in V_t$  whose degree is greater than  $t^{1-\varepsilon}$  for any positive small  $\varepsilon$ . We construct  $H_{2t}$  by connecting to  $v_*$  all vertices in  $V_t$  having degree greater than  $t^{2\varepsilon}$ , with  $\varepsilon < \gamma/4$ . To do this, let  $V_{t,\varepsilon}$  be the following random set:

$$V_{t,\varepsilon} = \{w \in V_t; d_t(w) \geq t^{2\varepsilon}\}.$$

Notice that  $v_* \in V_{t,\varepsilon}$ . Now, fix  $w \in V_{t,\varepsilon}$  and denote

$$h_s(w, v_*) := \mathbb{1}\{w \text{ connects to } v_* \text{ at step } s\}.$$

Moreover, let  $B_t$  be the event  $\{d_{\max}(G_t(q_\gamma)) > t^{1-\varepsilon}\}$ , i.e.,  $B_t$  is the event in which  $v_*$  actually exists. For  $t < s \leq 2t$  and some positive constant  $C$ , we have

$$\begin{aligned} \mathbb{E}[h_{s+1}(w, v_*) \mathbb{1}_{B_t} \mathbb{1}_{\{w \in V_{t,\varepsilon}\}} | \mathcal{F}_s] &= 1 - (1 - s^{-\gamma}) \frac{d_s(w) d_s(v_*)}{2s^2} \mathbb{1}_{B_t} \mathbb{1}_{\{w \in V_{t,\varepsilon}\}} \\ (5.1) \qquad \qquad \qquad &\leq 1 - \frac{C}{t^{1-\varepsilon}} \mathbb{1}_{B_t} \mathbb{1}_{\{w \in V_{t,\varepsilon}\}} \end{aligned}$$



since  $d_s(w) \geq t^{2\varepsilon}$ ,  $d_s(v_*) \geq t^{1-\varepsilon}$  and  $s \leq 2t$  in the above events. Therefore,

$$(5.2) \quad \mathbb{E} \left[ \prod_{s=t+1}^{2t} h_{s+1}(w, v_*) \mathbb{1}_{B_t} \mathbb{1}_{\{w \in V_{t,\varepsilon}\}} \middle| \mathcal{F}_t \right] \leq \left( 1 - \frac{C}{t^{1-\varepsilon}} \right)^t \mathbb{1}_{B_t} \mathbb{1}_{\{w \in V_{t,\varepsilon}\}} \\ \leq \exp \{ -Ct^\varepsilon \} \mathbb{1}_{B_t} \mathbb{1}_{\{w \in V_{t,\varepsilon}\}}.$$

By using the union bound on the vertices, we are able to get

$$(5.3) \quad \mathbb{P} \left( \bigcup_{w \in V_{t,\varepsilon}} \{w \text{ does not connect to } v_* \text{ in } G_{2t}\}, B_t \right) \leq t \exp \{ -Ct^\varepsilon \}.$$

So far we have proven that, *w.h.p.*, there exists a connected subgraph  $H_{2t}$  of  $G_{2t}$  which has  $V_{t,\varepsilon}$  as vertex set. To see why  $d_{2t}(H_{2t}) = t(1 - o(1))$ , first recall that  $|V_t|$  is highly concentrated around  $t^{1-\gamma}$ , by Lemma 1 and (2.5). Then observe that, whenever  $|V_t| \leq t^{1-\gamma/2}$ , we have

$$\sum_{v \in V_t, d_t(v) < t^{2\varepsilon}} d_t(v) \leq t^{1-\gamma/2} t^{2\varepsilon} = o(t),$$

as long as  $\varepsilon < \gamma/4$ . Since the sum of all degrees at time  $t$  equals  $2t$ , we have

$$\sum_{v \in V_{t,\varepsilon}} d_t(v) \geq 2t - t^{1-\gamma/2+2\varepsilon}.$$

The above discussion implies

$$\mathbb{P} \left( \sum_{v \in V_{t,\varepsilon}} d_t(v) < 2t - t^{1-\gamma/2+2\varepsilon} \right) \leq \mathbb{P} (|V_t| > t^{1-\gamma/2})$$

which is enough to conclude the proof.  $\square$

**Lemma 7.** *Let  $H$  be any attractive graph of  $G_t(q_\gamma)$ . Then, there exist positive constants  $c_1, c_2$  such that*

$$\mathbb{P} \left( \bigcup_{\substack{v \in V_t, \\ d_t(v) \geq c_1 \log(t)}} \{v \leftrightarrow H \text{ in } G_{2t}\} \right) \leq \frac{c_2}{t}.$$

*Proof.* Let  $A_t$  be the event where there exists an attractive graph  $H$  in  $G_t$ , so for some constant  $c_3 > 0$  we have  $d_t(H) \geq c_3 t$ . Analogously to the proof of Lemma 6, we let  $h_s(v, H)$  denote the indicator of the event where  $v$  does not connect to  $H$  at the  $s$ -th step. Notice

that, for  $s \in (t+1, 2t]$ , we have

$$\begin{aligned} \mathbb{E} [h_{s+1}(v, H) \mathbb{1}_{A_t} \mathbb{1}_{\{d_t(v) \geq c_1 \log(t)\}} | \mathcal{F}_s] &= \left( 1 - \frac{(1 - (s+1)^{-\gamma}) d_s(v) d_s(H)}{2s^2} \right) \mathbb{1}_{A_t} \mathbb{1}_{\{d_t(v) \geq c_1 \log(t)\}} \\ &\leq \left( 1 - \frac{(1 - (s+1)^{-\gamma}) c_1 \log(t) c_3 t}{8t^2} \right) \mathbb{1}_{A_t} \mathbb{1}_{\{d_t(v) \geq c_1 \log(t)\}} \\ &\leq \exp \left\{ -\frac{c_4 \log(t)}{t} \right\}. \end{aligned}$$

Taking the expected value on the above inequality, we obtain

$$(5.4) \quad \mathbb{P}(h_{s+1}(v, H), A_t, d_t(v) \geq c_1 \log(t)) \leq \exp \left\{ -\frac{c_4 \log(t)}{t} \right\}$$

which, after considering the intersection  $\bigcap_{s \in (t+1, 2t]} h_{s+1}(v, H)$ , leads to

$$(5.5) \quad \mathbb{P}(v \leftrightarrow H \text{ in } G_{2t}, d_t(v) \geq c_1 \log(t)) \leq \exp \{-c_4 \log(t)\}.$$

Choosing properly the constants  $c_1$  and  $c_2$  we are able to obtain

$$\mathbb{P}(v \leftrightarrow H \text{ in } G_{2t}, d_t(v) \geq c_1 \log(t)) \leq \frac{2}{t^2}.$$

Recalling that the cardinality of the random set  $\{v \in V_t | d_t(v) \geq c_1 \log(t)\}$  is bounded from above by  $t$ , the above upper bound combined with an union bound over  $v$  proves the Lemma.  $\square$

The proofs of the two final lemmas require some new definitions, which are given below. For a fixed time  $s_0 < t$  and a fixed positive constant  $c_1$ , consider the following random subsets of  $V_t$

$$(5.6) \quad R[s_0, t] := \{v_i \in V_t; i \in [s_0, t], d_t(v_i) < c_1 \log(t)\},$$

$$(5.7) \quad W_{s_0} := \{v_i \in V_t; i \in [1, s_0], d_t(v_i) < c_1 \log(t)\},$$

$$(5.8) \quad L_t := \{v_i \in V_t; d_t(v_i) \geq c_1 \log(t)\}$$

and finally

$$(5.9) \quad V[s, t] := \{v_i \in V_t; i \in [s, t]\}.$$

**Lemma 8.** *Using the above definitions, we have*

$$\mathbb{P} \left( \max_{u \in R[t^{1/3}, t]} \text{dist}(u, L_t) > \frac{3}{\gamma} \right) = o(1).$$

*Proof.* Fix  $K \in \mathbb{N}$  and for each integer sequence  $t_1 < t_2 \cdots < t_K$ , with  $t_i \in [t^{1/3}, t]$ . Recall the definition of the uniform random variables  $(U_t)_{t \geq 1}$  from Section 2. We denote by  $\{v_{t_i} \rightarrow v_{t_{i-1}}\}$

the event where the vertex born at time  $t_i$  makes its first connection to the vertex born at time  $t_{i-1}$ . We then let  $A_{t_1, \dots, t_K}$  denote the following event

$$(5.10) \quad A_{t_1, \dots, t_K} := \{U_{t_1} \leq t_1^{-\gamma}\} \bigcap_{i=2}^K (\{U_{t_i} \leq t_i^{-\gamma}\} \cap \{v_{t_i} \rightarrow v_{t_{i-1}}\} \cap \{d_{t_{i-1}}(v_{t_{i-1}}) \leq c_1 \log(t)\}).$$

Recall Theorem 3.15 which guarantees that *w.h.p* all vertices added before  $t^{1/3}$  reach degree at least  $c_1 \log(t)$  provided  $c_1$ . Therefore,  $W_{t^{1/3}}$  is empty *w.h.p*. In this case, the only way we may have, for some  $u \in R[t^{1/3}, t]$ ,  $\text{dist}(u, L_t) > K$  is by finding a path of length at least  $K$  whose vertices, except for the first one, belong to  $R[t^{1/3}, t]$ . This affirmation may be summarized in the following inclusion of events

$$(5.11) \quad \left\{ \max_{u \in R[t^{1/3}, t]} \text{dist}(u, L_t) > K \right\} \cap \{W_t = \emptyset\} \subset \bigcup_{t^{1/3} \leq t_1 < \dots < t_K \leq t} A_{t_1, \dots, t_K}.$$

Thus, the Lemma is proven if we guarantee a small enough upper bound on the probability of the union of  $A_{t_1, \dots, t_K}$ 's. Observe that, for each such event, we have the upper bound below

$$(5.12) \quad \mathbb{P}(A_{t_1, \dots, t_K}) \leq \frac{c_1^K \log^K(t)}{t_1^\gamma \dots t_K^\gamma (t_2 - 1) \dots (t_K - 1)} \leq \frac{c_1^K \log^K(t)}{t_1^\gamma (t_2 - 1)^{1+\gamma} \dots (t_K - 1)^{1+\gamma}}.$$

Each vertex  $v_{t_i}$  on the path is created with probability  $f(t_i) = t_i^{-\gamma}$ , explaining the product  $(t_1 \dots t_K)^{-\gamma}$ . At time  $t_i$ ,  $v_{t_i}$  connects to  $v_{t_{i-1}}$  whose degree at time  $t_i - 1$  is at most  $c_1 \log(t)$ .

Summing over all possible choices for the  $t_i$ 's, we obtain the following

$$(5.13) \quad \begin{aligned} \mathbb{P} \left( \bigcup_{t^{1/3} \leq t_1 < \dots < t_K \leq t} A_{t_1, \dots, t_K} \right) &\leq \sum_{t^{1/3} \leq t_1 < t_2 < \dots < t_K \leq t} \frac{c_1^K \log^K(t)}{t_1^\gamma t_2^{1+\gamma} \dots t_K^{1+\gamma}} \\ &\leq c_1^K \log^K(t) \sum_{t_1 = t^{1/3}}^t t_1^{-\gamma} \left( \sum_{t_2 = s_0}^t t_2^{-1-\gamma} \right)^{K-1} \\ &\leq C \frac{\log^K(t) t^{1-\gamma}}{t^{(K-1)\gamma/3}}. \end{aligned}$$

Taking  $K = 3/\gamma$ , the above upper bound is  $o(1)$ . Since  $\mathbb{P}(W_{t^{1/3}} \neq \emptyset) = o(1)$  by Theorem 3.15, the Lemma follows.  $\square$

**Lemma 9.** *Throughout this lemma, we let  $\text{dist}$  refer to the graph distance of  $G_{2t}(q_\gamma)$ . We have*

$$\mathbb{P} \left( \max_{u \in V_{2t} \setminus V_t} \text{dist}(u, V_t) > \frac{3}{\gamma} \right) = o(1).$$

*Proof.* As usual, we start with a definition. For  $s \in [t, 2t]$ , let  $B_s$  be the event below

$$(5.14) \quad B_s := \bigcup_{v \in V_s \setminus V_t} \left\{ d_s(v) \geq Ct^{\frac{1}{2N}} \log^N(t) \right\},$$

where  $N, M$  are positive integers which will be chosen properly later. In words,  $B_s$  denotes the event in which all vertices of  $G_s(q_\gamma)$  added after time  $t$  have degree at least a small power of  $t$  times a large power of  $\log(t)$ . Observe that, by Proposition 3.6, we have  $\mathbb{P}(B_s) \leq N/t^M$ .

Set  $K = 3/\gamma$ , and for each sequence of times  $t \leq t_1 < t_2 < \dots < t_K \leq 2t$ , let  $A_{t_1, \dots, t_K}$  be

$$(5.15) \quad A_{t_1, \dots, t_K} := \bigcap_{i=2}^K \left( \{U_{t_i} \leq t_i^{-\gamma}\} \cap \{v_{t_i} \rightarrow v_{t_{i-1}}\} \right) \cap \{U_{t_1} \leq t_1^{-\gamma}\}.$$

Since  $G_{2t}(q_\gamma)$  is connected, we have the following inclusion of events

$$\left\{ \max_{u \in V_{2t} \setminus V_t} d_{G_{2t}}(u, V_t) > K \right\} \subset \bigcup_{t \leq t_1 < \dots < t_K \leq 2t} A_{t_1, \dots, t_K}.$$

Now we provide a useful upper bound for the probability of  $A_{t_1, \dots, t_K}$  in the following way. Observe that

$$(5.16) \quad \mathbb{P}(v_{t_K} \rightarrow v_{t_{K-1}} | U_{t_K} \leq t_K^{-\gamma}, A_{t_1, \dots, t_{K-1}}, B_{t_{K-1}}^c) \leq \frac{Ct^{\frac{1}{2N}} \log(t)^N}{t_K - 1}$$

therefore,

$$(5.17) \quad \mathbb{P}(A_{t_1, \dots, t_K}, B_{t_{K-1}}^c) \leq \frac{Ct^{\frac{1}{2N}} \log(t)^N}{t_K - 1} \cdot \frac{1}{t_K^\gamma} \mathbb{P}(A_{t_1, \dots, t_{K-1}})$$

which implies

$$(5.18) \quad \mathbb{P}(A_{t_1, \dots, t_K}) \leq \frac{Ct^{\frac{1}{2N}} \log(t)^N}{t_K - 1} \frac{1}{t_K^\gamma} \mathbb{P}(A_{t_1, \dots, t_{K-1}}) + \mathbb{P}(B_{t_{K-1}}).$$

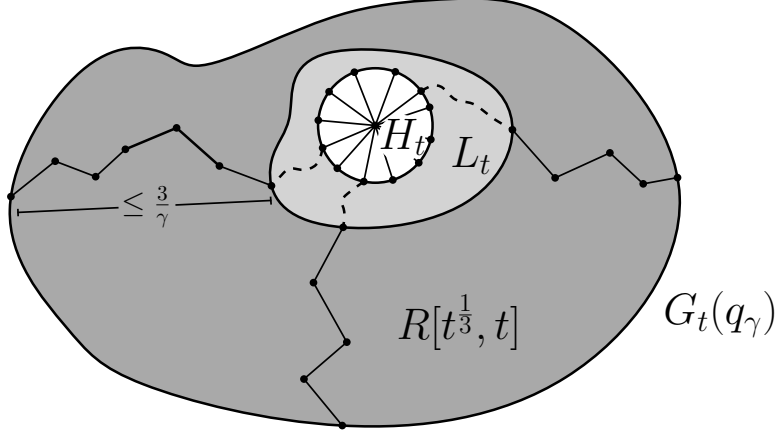
Repeating this procedure and recalling that  $t \leq t_i \leq 2t$ , we may deduce

$$(5.19) \quad \begin{aligned} \mathbb{P}(A_{t_1, \dots, t_K}) &\leq \frac{C^K t^{\frac{K}{2N}} \log^{KN}(t)}{t_1^\gamma [(t_2 - 1)(t_3 - 1) \dots (t_K - 1)]^{1+\gamma}} + \frac{KN}{t^M} \\ &\leq \frac{C' t^{\frac{K}{2N}} \log^{KN}(t)}{t^{K(1+\gamma)-1}} + \frac{KN}{t^M} \end{aligned}$$

which leads to

$$(5.20) \quad \mathbb{P}\left(\bigcup_{t \leq t_1, \dots, t_K \leq 2t} A_{t_1, \dots, t_K}\right) \leq \frac{C' t^{K + \frac{K}{2N}} \log^{KN}(t)}{t^{K+K\gamma-1}} + \frac{KN}{t^{M-K}}.$$

Finally, choosing  $M > K$ ,  $N$  large enough so that  $1/2^N < \gamma/2$  and recalling that  $K = 3/\gamma$ , we obtain the desired result.  $\square$

FIGURE 5. Picture of a *good*  $G_t(q_\gamma)$ .

Now we have all the tools needed for the proof of the section's main result.

**Theorem 6** (The upper bound for the diameter). *For all  $\gamma \in (0, 1)$ ,*

$$\mathbb{P} \left( \text{diam}(G_{2t}(q_\gamma)) > 4 + \frac{6}{\gamma} + \frac{6}{\gamma} \right) = o(1).$$

*Proof.* Call  $G_t(q_\gamma)$  *good* if and only if it contains one attractive graph  $H_t$  with diameter 2, all the vertices added before time  $t^{1/3}$  have degree at least  $c_1 \log(t)$ , i.e.,  $W_{t^{1/3}} = \emptyset$ , and finally for all  $u \in R[t^{1/3}, t]$  we have  $\text{dist}(u, L_t) \leq \frac{3}{\gamma}$ . Otherwise, call it *bad*. By Lemma 6, Proposition 3.15 and Lemma 8,  $G_t(q_\gamma)$  is good *w.h.p.* When this is the case, the picture we have of  $G_t(q_\gamma)$  is similar to Figure 5.

Now, let  $B_{2t}$  be the following event

$$B_{2t} := \bigcup_{v \in L_t} \{v \leftrightarrow H_t, \text{ in } G_{2t}(q_\gamma)\}.$$

By Lemma 7,  $B_{2t}$  is very unlikely. We must observe that if we are in  $\{G_t(q_\gamma) \text{ is good}\}$  and  $B_{2t}^c$  then  $G_t(q_\gamma)$  seen as a subgraph of  $G_{2t}(q_\gamma)$  has diameter at most  $4 + 6/\gamma$ , since, in this case, all vertices in  $L_t$  connect to  $H_t$  producing a subgraph  $H'_t$  of diameter at most 4. Moreover, since  $G_t(q_\gamma)$  is good, the distance of  $v \in V_t \setminus H'_t$  from  $H'_t$  is bounded by  $3/\gamma$ . Hence, in order for  $\text{diam}(G_{2t}(q_\gamma)) \geq 4 + \frac{6}{\gamma} + \frac{6}{\gamma}$  there must exist some vertex in  $V_{2t} \setminus V_t$  whose distance from  $V_t$  is greater than  $3/\gamma$ , which is very unlikely according to Lemma 9. Otherwise  $\text{diam}(G_{2t}(q_\gamma)) \leq 4 + \frac{6}{\gamma} + \frac{6}{\gamma}$ . Let  $A_{2t}$  denote the event  $\{\text{diam}(G_{2t}(q_\gamma)) > 4 + \frac{6}{\gamma} + \frac{6}{\gamma}\}$ . Then,

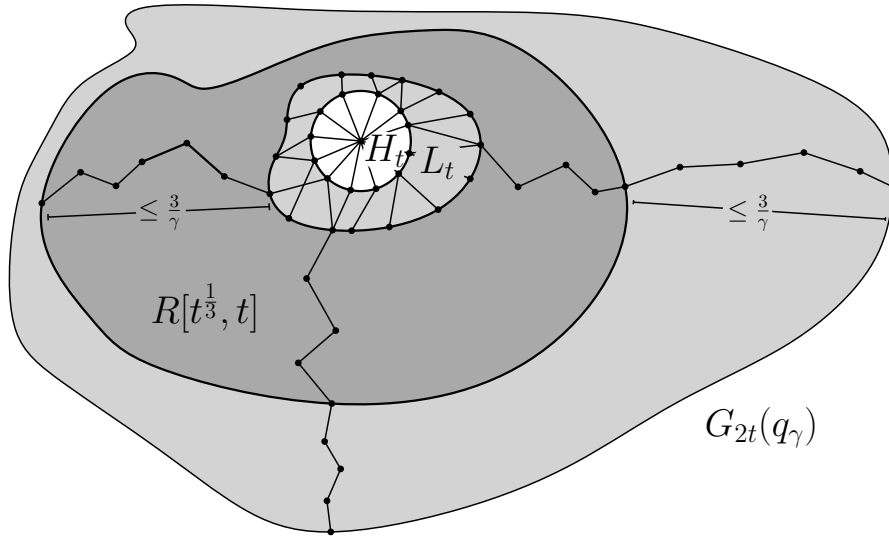


FIGURE 6. After separating  $G_{2t}(q_\gamma)$  into regions that we subsequently study, we are able to show a constant upper bound for  $\text{diam}(G_{2t}(q_\gamma))$  w.h.p..

we have

$$\begin{aligned}
 \mathbb{P}(A_{2t}) &\leq \mathbb{P}(A_{2t}, G_t(q_\gamma) \text{ is good}) + \mathbb{P}(G_t(q_\gamma) \text{ is bad}) \\
 &\leq \mathbb{P}(A_{2t}, B_{2t}^c, G_t(q_\gamma) \text{ is good}) + \mathbb{P}(B_{2t}) + \mathbb{P}(G_t(q_\gamma) \text{ is bad}) \\
 &\leq \mathbb{P}\left(\max_{u \in V_{2t} \setminus V_t} \text{dist}(u, V_t) > \frac{3}{\gamma}\right) + \mathbb{P}(B_{2t}) + \mathbb{P}(G_t(q_\gamma) \text{ is bad}).
 \end{aligned}$$

And by Lemma 9, Lemma 6, Proposition 3.15 and Lemma 8 all the probabilities involved in the above upper bound are  $o(1)$ , proving the theorem.  $\square$

## APPENDIX A. DEGREE DISTRIBUTION

We let  $N_t(d, f)$  denote both the number of vertices of degree  $d$  in  $G_t(f)$  for a generic edge-step function  $f$ . When the function  $f$  is clear from the context, we will omit it from the notation.

**Lemma 10** (Lemma 3.1 of [4]). *Let  $a_t$  be a sequence of positive real numbers satisfying the recurrence relation*

$$a_{t+1} = \left(1 - \frac{b_t}{t}\right) a_t + c_t.$$

*Furthermore, suppose  $b_t \rightarrow b > 0$  and  $c_t \rightarrow c$ . Then*

$$\lim_{t \rightarrow \infty} \frac{a_t}{t} = \frac{c}{1+b}.$$

**Lemma 11.** *We have that  $\mathbb{E}N_t(d) = \mathbb{E}N_t(d, f)$  satisfies*

$$(A.1) \quad \mathbb{E}N_t(1) = \left(1 - \frac{2 - f(t)}{2(t-1)}\right) \mathbb{E}N_{t-1}(1) + f(t) + O\left(\frac{\mathbb{E}V_t}{t^2}\right),$$

and for a fixed integer  $d \geq 2$ ,

$$(A.2) \quad \mathbb{E}[N_t(d)] = \left(1 - \frac{(2 - f(t))d}{2(t-1)}\right) \mathbb{E}[N_{t-1}(d)] + \frac{(d-1)(2 - f(t))}{2(t-1)} \mathbb{E}[N_{t-1}(d-1)] + d^2 O\left(\frac{\mathbb{E}V_{t-1}}{t^2}\right).$$

*Proof.* Observe that for a generic vertex  $v$ , the models previously defined satisfy the following equations involving the increment of vertex  $v$ 's degree

$$(A.3) \quad \begin{aligned} \mathbb{P}(\Delta d_t(v) = 1 | \mathcal{F}_t) &= f(t+1) \frac{d_t(v)}{2t} + 2(1 - f(t+1)) \frac{d_t(v)}{2t} \left(1 - \frac{d_t(v)}{2t}\right) \\ &= \left(1 - \frac{f(t+1)}{2}\right) \frac{d_t(v)}{t} - 2(1 - f(t+1)) \frac{d_t^2(v)}{4t^2}, \end{aligned}$$

$$(A.4) \quad \mathbb{P}(\Delta d_t(v) = 2 | \mathcal{F}_t) = (1 - f(t+1)) \frac{d_t^2(v)}{4t^2}.$$

Moreover, for a fixed degree  $d$ , we may write  $N_{t+1}(d)$  as

$$(A.5) \quad N_{t+1}(d) = \sum_{\substack{v \in G_t(f) \\ d_t(v)=d}} \mathbb{1}\{\Delta d_t(v) = 0\} + \sum_{\substack{v \in G_t(f) \\ d_t(v)=d-1}} \mathbb{1}\{\Delta d_t(v) = 1\} + \sum_{\substack{v \in G_t(f) \\ d_t(v)=d-2}} \mathbb{1}\{\Delta d_t(v) = 2\}.$$

Combining the three above equations and taking the expected value on (A.5), we obtain (A.2). For the case  $d = 1$ , just observe that

$$N_{t+1}(1) = \sum_{\substack{v \in G_t(f) \\ d_t(v)=d}} \mathbb{1}\{\Delta d_t(v) = 0\} + \mathbb{1}\{\text{a vertex is born at time } t+1\}$$

The term  $d^2 O\left(\frac{\mathbb{E}V_t}{t^2}\right)$  arises after we isolate the terms of quadratic order of  $t$  in the expected value of  $N_{t+1}(d)$  and from the fact that  $N_t(d) \leq V_t$  for all  $d$ . Observe that  $O\left(\frac{\mathbb{E}V_t}{t^2}\right) \leq O(t^{-1})$ , since  $V_t \leq t$ .  $\square$

We now prove two results that will imply Proposition 1, which stated that the degree distributions of the random graphs with edge step function given by  $\ell_M$  and  $q_\gamma$  obey a power-law distribution with exponents 2 and  $2 - \gamma$ , respectively.

**Claim 2.** *Let  $N_t(d)$  be the number of vertices of degree  $d$  in the graph  $G_t(\ell_M)$ . Then we have*

$$\frac{\mathbb{E}N_t(d)}{t(\log(t))^{-M}} \xrightarrow{t \rightarrow \infty} \frac{1}{3(d+1)d}$$

*Proof of the claim:* By Equation (2.4), we know that

$$\mathbb{E}V_t = \Theta\left(\frac{t}{(\log(t))^M}\right).$$

Furthermore, since we know that  $V_t$  is concentrated around its expected value, we have that in order to calculate the expected proportion of vertices of degree  $d$ , one must divide  $\mathbb{E}N_t(d)$  by a function of order  $t(\log(t))^{-M}$ .

To simplify our writing, let  $a_t(d)$  denote  $\mathbb{E}N_t(d)$ . Then, equation (A.2) becomes (A.6)

$$a_{t+1}(d) = \left(1 - \frac{(2 - \ell_M(t+1))d}{2t}\right) a_t(d) + \frac{(d-1)(2 - \ell_M(t+1))}{2t} a_t(d-1) + O\left(\frac{1}{t(\log(t))^M}\right).$$

From now on we will suppress the term of order  $1/t(\log(t))^M$  present in the above equation, since it will play no part when we apply Lemma 10. Now, we do the following substitution on the above equation

$$a'_t(d) = a_t(d)(\log(t))^M,$$

for every  $d \geq 1$ . This leads to the following recurrence relation involving  $a'_t(d)$ :

$$\begin{aligned} a'_{t+1}(d) &= \left(1 - \frac{(2 - \ell_M(t+1))d}{2t}\right) \left(\frac{\log(t+1)}{\log(t)}\right)^M a'_t(d) \\ &\quad + \left(\frac{\log(t+1)}{\log(t)}\right)^M \frac{(d-1)(2 - \ell_M(t+1))}{2t} a'_t(d-1). \end{aligned}$$

Now, observe that as  $t$  gets large we may write

$$\left(\frac{\log(t+1)}{\log(t)}\right)^M = 1 + \frac{M}{t \log(t)} + O(t^{-2}),$$

which yields

$$\begin{aligned} (A.7) \quad a'_{t+1}(d) &= \left(1 - \frac{(2 - \ell_M(t+1))d}{2t} + O\left(\frac{1}{t \log(t)}\right)\right) a'_t(d) \\ &\quad + \left(\frac{\log(t+1)}{\log(t)}\right)^M \frac{(d-1)(2 - \ell_M(t+1))}{2t} a'_t(d-1). \end{aligned}$$

We conclude the proof proceeding by induction on  $d$ . Assume, for  $k \leq d-1$ , that

$$(A.8) \quad \frac{a_t(k)}{t(\log(t))^{-M}} = \frac{a'_t(k)}{t} \xrightarrow{t \rightarrow \infty} M_k$$

where  $M_k$  is a positive number depending on  $d$ . Then, using Lemma 10 and the inductive hypothesis we obtain

$$\frac{\mathbb{E}[N_t(d)]}{t(\log(t))^{-M}} = \frac{a'_t(d)}{t} \xrightarrow{t \rightarrow \infty} \frac{(d-1)}{d+1} M_{d-1}.$$



The Claim will then be proven when we prove (A.8) for  $d = 1$ . By (A.1), we know that

$$(A.9) \quad a_{t+1}(1) = \left(1 - \frac{(2 - \ell_M(t+1))}{2t}\right) a_t(1) + \ell_M(t+1) + O\left(\frac{1}{t(\log(t))^M}\right).$$

Using that  $a'_t(1) = (\log(t))^M a_t(1)$  we obtain

$$a'_{t+1}(1) = \left(1 - \frac{2 - \ell_M(t+1)}{2t} + O\left(\frac{1}{t \log(t)}\right)\right) a'_t(1) + (\log(t+1))^M \ell_M(t+1) + O\left(\frac{1}{t}\right).$$

Finally, using Lemma 10 with  $b_t = 1 - \ell_M(t+1)/2$  and  $c_t = 1 + O(t^{-1})$  we prove the first step of the induction with  $M_1 = 3^{-1}$ . The claim then follows once one observes that

$$\left(\frac{d-1}{d+1}\right) \left(\frac{d-2}{d}\right) \left(\frac{d-3}{d-1}\right) \cdots \left(\frac{2}{4}\right) \left(\frac{1}{3}\right) M_1 = \frac{1}{3(d+1)d}.$$

■

Now we focus on the case  $q_\gamma$ .

**Claim 3.** *Let  $N_t(d)$  be the number of vertices of degree  $d$  in the graph  $G_t(q_\gamma)$ , with  $\gamma \in (0, 1)$ . There exists a positive constant  $c(\gamma)$  such that*

$$\frac{\mathbb{E}N_t(d)}{t^{1-\gamma}} \xrightarrow{t \rightarrow \infty} \frac{c(\gamma)}{d^{2-\gamma}}.$$

*Proof of the claim:* We proceed as in the case for  $\ell_M$ . We know by (2.5) that

$$\mathbb{E}V_t = \Theta(t^{1-\gamma})$$

and that, by Lemma 1, the random variable  $V_t$  concentrates around its mean. With that in mind, we again denote  $\mathbb{E}V_t$  by  $a_t(d)$  and then perform the following substitution in order to apply Lemma 10

$$a'_t(d) = t^\gamma a_t(d),$$

which yields

$$(A.10) \quad \begin{aligned} a'_{t+1}(d) &= \left(1 - \frac{(2 - q_\gamma(t+1))d}{2t}\right) \left(1 + \frac{1}{t}\right)^\gamma a'_t(d) \\ &\quad + \left(1 + \frac{1}{t}\right)^\gamma \frac{(d-1)(2 - q_\gamma(t+1))}{2t} a'_t(d-1) + O(t^{-1}). \end{aligned}$$

Now, observe that as  $t$  gets large we may write

$$\left(1 + \frac{1}{t}\right)^\gamma = 1 + \frac{\gamma}{t} + O(t^{-2}).$$

Returning to (A.10), we have

$$(A.11) \quad \begin{aligned} a'_{t+1}(d) &= \left(1 - \frac{(2 - q_\gamma(t+1))d - 2\gamma}{2t}\right) a'_t(d) \\ &\quad + \frac{(d-1)(2 - q_\gamma(t+1))}{2t} a'_t(d-1) + O(t^{-1}). \end{aligned}$$

We again proceed by induction. We assume that, for  $k \leq d - 1$ ,

$$(A.12) \quad \frac{a_t(k)}{t^{1-\gamma}} = \frac{a'_t(k)}{t} \xrightarrow{t \rightarrow \infty} M_k.$$

Using Lemma 10, equation (A.11), and the inductive hypothesis, we obtain

$$\frac{a_t(d)}{t^{1-\gamma}} = \frac{a'_t(d)}{t} \xrightarrow{t \rightarrow \infty} \frac{(d-1)}{d+1-\gamma} M_{d-1}.$$

To trigger the induction on  $d$  we recall the recurrence relation for  $d = 1$  (A.1),

$$(A.13) \quad a_{t+1}(1) = \left(1 - \frac{(2 - q_\gamma(t+1))}{2t}\right) a_t(1) + q_\gamma(t+1) + O(t^{-1-\gamma}),$$

and the substitution  $a'_t(1) = t^\gamma a_t(1)$  in order to obtain

$$a'_{t+1}(1) = \left(1 - \frac{2 - q_\gamma(t+1) - 2\gamma}{2t}\right) a'_t(1) + (t+1)^\gamma q_\gamma(t+1) + O(t^{-1}).$$

Finally, applying Lemma 10 with  $b_t = 1 - \gamma - q_\gamma(t+1)/2$  and  $c_t = 1 + O(t^{-1})$  we prove the first induction step. To finish the proof of the claim, we note that

$$\frac{\mathbb{E} N_t(d)}{t^{1-\gamma}} \xrightarrow{t \rightarrow \infty} C_1 \prod_{k=2}^d \frac{k-1}{k+1-\gamma} = C_2 \frac{\Gamma(d)}{\Gamma(d+2-\gamma)} = \Theta\left(\frac{1}{d^{2-\gamma}}\right).$$

■

We are now able to finish the proof of Theorem 1:

*Proof of Theorem 1.* As previously stated, we know by Lemma 1 that both  $|V(G_t(\ell_M))|$  and  $|V(G_t(q_\gamma))|$  concentrate around their means. Equations (2.4) and (2.5) together with Claims 2 and 3 then imply

$$\frac{\mathbb{E}[N_t(d, \ell_M)]}{\mathbb{E}[V_t(G_t(\ell_M))]} \xrightarrow{t \rightarrow \infty} \Theta\left(\frac{1}{d^2}\right) \quad \text{and for } \gamma < 1, \quad \frac{\mathbb{E}[N_t(d, q_\gamma)]}{\mathbb{E}[V_t(G_t(q_\gamma))]} \xrightarrow{t \rightarrow \infty} \Theta\left(\frac{1}{d^{2-\gamma}}\right).$$

This finishes the proof of the result. □

**A.1. The case  $q_1$ .** We treat the case  $f(t) = 1/t$  separately since its degree distribution does not obey a power law. One should note that, by Theorem 3, there exists with high probability a complete subgraph of  $G_t(q_1)$  with size  $(1/2 - \delta) \log(t)$  for any given  $\delta > 0$ . However, in this case,  $|V_t|$  has order  $\log(t)$ , as one can notice by simple integration and Lemma 1. Thus, this implies that actually a positive fraction of vertices have degree at least  $(1/2 - \delta) \log(t)$ , *w.h.p.*

*Proof of Theorem 2.* Similarly to the proof of Claim 2, we proceed by induction on  $d$ . We begin by noticing that  $\mathbb{E} V_t = \Theta(\log(t))$ , since  $q_1(t) = 1/t$ . Thus, according to Lemma 11, we

have, for  $d = 1$

$$(A.14) \quad a_{t+1}(1) = \left(1 - \frac{1}{t} + \frac{1}{2t^2}\right) a_t(1) + \frac{1}{t+1} + O\left(\frac{\log(t)}{t^2}\right).$$

Instead of applying Lemma 10, we expand the above recurrence relation to obtain

$$(A.15) \quad \begin{aligned} a_{t+1}(1) &= \frac{1}{t+1} + \sum_{s=1}^t \left( \frac{1}{s} + O\left(\frac{\log(s-1)}{(s-1)^2}\right) \right) \left( \prod_{r=s}^t \left( 1 - \frac{1}{r} + \frac{1}{2r^2} \right) \right) \\ &\leq \frac{1}{t+1} + \exp\left\{ \sum_{r=1}^{\infty} \frac{1}{2r^2} \right\} \sum_{s=1}^t \left( \frac{1}{s} + O\left(\frac{\log(s-1)}{(s-1)^2}\right) \right) \exp\left\{ -\sum_{r=s}^t \frac{1}{r} \right\} \\ &\leq \frac{1}{t+1} + c_1 \sum_{s=1}^t \frac{s}{t} \cdot \frac{1}{s} = c_2 + \frac{1}{t+1} \leq C_1. \end{aligned}$$

Now, assume that for  $k \leq d-1$  there exists a positive number  $C_k$  such that  $a_t(k) \leq C_k$  and recall the recurrence relation given by (A.2), which gives us

$$a_{t+1}(d) = \left(1 - \frac{d}{t} + \frac{d}{2t^2}\right) a_t(d) + \left(\frac{d-1}{t} - \frac{d-1}{2t^2}\right) a_t(d-1) + O\left(\frac{\log(t)}{t^2}\right).$$

Expanding the above equality, we obtain

$$(A.16) \quad \begin{aligned} a_{t+1}(d) &= \sum_{s=1}^t \left( \left( \frac{d-1}{s} - \frac{d-1}{2s^2} \right) a_s(d-1) + O\left(\frac{\log(s-1)}{(s-1)^2}\right) \right) \prod_{r=s}^t \left( 1 - \frac{d}{r} + \frac{d}{2r^2} \right) \\ &\leq \sum_{s=1}^t c_1 \frac{s^d}{t^d} \frac{c_2}{s} a_s(d-1) \\ &\leq \sum_{s=1}^t c_3 \frac{s^{d-1}}{t} \leq C_d, \end{aligned}$$

where we used the inductive hypothesis which gives us a universal upper bound for  $a_s(d-1)$ . Finally, dividing  $a_t(d)$  by  $\log(t)$  we have the desired result.  $\square$

## APPENDIX B. MARTINGALES CONCENTRATION INEQUALITIES

For the sake of completeness we state here two useful concentration inequalities for martingales which are used throughout the paper.

**Theorem B.1** (Azuma-Höfeding Inequality - [4]). *Let  $(M_n, \mathcal{F})_{n \geq 1}$  be a (super)martingale satisfying*

$$|M_{i+1} - M_i| \leq a_i$$

*Then, for all  $\lambda > 0$  we have*

$$\mathbb{P}(M_n - M_0 > \lambda) \leq \exp\left(-\frac{\lambda^2}{\sum_{i=1}^n a_i^2}\right).$$

**Theorem B.2** (Freedman's Inequality - [9]). *Let  $(M_n, \mathcal{F}_n)_{n \geq 1}$  be a (super)martingale. Write*

$$V_n := \sum_{k=1}^{n-1} \mathbb{E}[(M_{k+1} - M_k)^2 | \mathcal{F}_k]$$

*and suppose that  $M_0 = 0$  and*

$$|M_{k+1} - M_k| \leq R, \text{ for all } k.$$

*Then, for all  $\lambda > 0$  we have*

$$\mathbb{P}(M_n \geq \lambda, V_n \leq \sigma^2, \text{ for some } n) \leq \exp\left(-\frac{\lambda^2}{2\sigma^2 + 2R\lambda/3}\right).$$

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